GOVERNMENT OF THE REPUBLIC OF INDONESIA

MINISTRY OF PUBLIC WORKS

DIRECTORATE GENERAL OF WATER RESOURCES DEVELOPMENT

PROGRAMME OF ASSISTANCE FOR THE IMPROVEMENT OF HYDROLOGIC DATA COLLECTION, PROCESSING AND EVALUATION IN INDONESIA

BED-MATERIAL LOAD

(EINSTEIN'S METHOD)

by

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SOCIETE CENTRALE POUR L' EQUIPEMENT DU TERRITOIRE INTERNATIONAL

BANDUNG MARCH 1981

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LIST OF SYMBOLS

A	:	cross-sectional area
d	:	diameter of particle. In a mixture $d = d_{50}$ or median diameter
D	:	depth of flow
g	:	gravitational constant, mean value 9.81 m/s^2
g_s	:	bedload rate in weight per unit time and unit width
g_{ss}	:	suspended load rate in weight per unit time and unit width
g_{st}	:	bed-material load rate in weight per unit time and unit width
G_s	:	bedload rate in weight per unit time
G_{ss}	:	suspended load rate in weight per unit time
G_{st}	:	bed-material load rate in weight per unit time
n	:	Manning roughness value
p	:	fraction of bed material in a given grain size
P	:	wetted perimeter
Q	:	water discharge (m^3/s)
R_H	:	hydraulic radius $R_H = \frac{A}{P}$
S	:	channel slope
u	:	fluid velocity
u_*	:	shear or friction velocity $u_* = \sqrt{g R_H S}$
v	:	settling velocity of particle
ρ	:	density of fluid. For water at 20°C ρ = 1000 kg/m^3
ρ_s	:	density of particle. Usually taken as 2650 kg/m^3 when the actual value is unknown
γ	:	fluid specific weight. Water at 20°C = 1000 kgf/m^3
γ_s	:	particle specific weight. Taken usually as 2650 kgf/m^3 when actual value is unknown
ν	:	kinematic viscosity of fluid. For water at 20°C. $\nu = 10^{-2} cm^2/s$
τ_y	:	shear stress or friction force per unit area exerted by the fluid at a depth y above the bed
τ_o	:	shear stress at the bottom $\tau_o = \gamma R_H S$ or $\tau_o = \gamma DS$

Other symbols are defined in due course in the following sections.

INTRODUCTION

The bed-material load is made up of only those particles consisting of grain sizes represented in the bed.

In theory the bed-material load can be predicted with the hydraulic knowledge of the stream, that is,

- velocity
- bed composition and configuration
- shape of the measuring section
- water temperature
- concentration of fine sediment

Therefore the problem at issue is to determine the relationship between the bed-material load and the prevailing hydraulic conditions such a problem has proved to be a difficult task and is not yet completely solved.

So far comparisons of measured and calculated bed-material loads exhibit discrepancies which lead to think that first the problem of sediment transport is not fully understood and second great care must be taken in using bed-material load formulae.

As pointed out by GRAF (see references at the end) "Einstein's method represents the most detailed and comprehensive treatment, from the point of fluid mechanics, that is presently available". This method is described in the following paragraphs.

Nota : We prefer the name "Bed-material load" to the name "Total load" since the so-called "washload" is not taken into account when one speaks of bed-material load.

EINSTEIN'S PROCEDURE

Introduction

The bed-material load is divided in two parts according to the mode of transport. In the immediate vicinity of the bed in the so-called bed layer takes place the bedload whereas the suspended-load takes place above the bed layer where the particle's weight is supported by the surrounding fluid and thus the particles move with the flow at the same average velocity.

Some researchers think the division of the bed-material load in two fractions is questionable. Actually such a division is rather artificial particularly when it comes to define a zone of demarcation between bed-load and suspended-load, nevertheless it is often convenient for the sake of clarity to distinguish these two modes of transport.

Nota : Figures number 2 to number 9 are grouped from page 15 to page 20.

1. HYDRAULIC CALCULATIONS

1.1 Test Reach

To calculate or measure the flow and the sediment transport in a stream, a test reach has to be selected first, the following requirements have to be fulfilled, the better they are the more reliable the results.

- It should be sufficiently long to determine rather accurately the slope of the channel
- It should have a fairly uniform and stable channel geometry with uniform flow conditions and bed material composition
- It should have a minimum of outside effects such as strong bends, islands, sills or excessive vegetation
- No tributaries should join the river within or immediatly above the test reach.

It is worth noting that the foregoing requirements are those usually sought-for to set up a gauging station.

1.2 Surface Drag and Bed-Form Drag (or Bar Resistance)

To take into account the contribution the bedforms make to the channel roughness it was proposed that both the cross section area, denoted A, and the hydraulic radius, denoted R_H, be divided into two parts: one related to the surface drag or grain roughness designated by A' and R'_H , the other related to the bedform drag designated by A" and R''_H respectively.

In terms of hydraulic radii we have

$$R_H = R'_H + R''_H$$

It follows that both shear stress and friction velocity are in turn divided since:

$$\tau_o = \gamma R_H S = \gamma (R'_H + R'')S \quad \text{and} \tag{1}$$

$$u_* = \sqrt{g R_H S} = \sqrt{g(R'_H + R''_H)S} \tag{2}$$

so we have:

a. In terms of shear stresses

$$\tau_o = \tau'_o + \tau''_o \tag{3}$$

b. in terms of friction velocities

$$u_*^2 = u_*'^2 + u_*''^2 \tag{4}$$

the "prime", ' , used in the notation pertains to the surface drag whereas the "double prime", " , pertains to the bedform drag.

Einstein and Barbarossa derived a curve from data of river measurements which relates the "flow intensity" denoted Ψ_{35} and defined as

$$\Psi_{35} = \frac{\rho_s - \rho}{\rho} \frac{d_{35}}{R'_H S} \quad (d_{35} \text{ is the bed sediment size for which 35\% of the material is finer}) \tag{5}$$

to the ratio $\frac{\bar{u}}{u''_*}$ of the mean stream velocity, denoted $\bar{u}$, to the friction velocity due to the bar resistance denoted u''_*. This curve which has come to be known as "bar resistance curve" is shown in fig. 3.

Nota: Different bedform shapes are sketched in Annex 1

1.3 Mean Velocity

Depending on the surface roughness, Einstein and Barbarossa recommended use of either the Manning-Strickles equation or a logarithmic type formula.

a. Manning-Strickler's equation

Is defined as :

$$\frac{\bar{u}}{u'_*} = 7.66 \left(\frac{R'_H}{d_{65}}\right)^{1/6} \tag{6}$$

where d_{65} is the bed sediment size for which 65% of the bed material is finer.

The well-known Manning formula is defined as

$$\bar{u} = \frac{1}{n} R_H^{3/2} S^{1/2} \tag{7}$$

Let us assume firstly the velocity would be the same with a flat bed and secondly the bedform would affect both the roughness coefficient n and the hydraulic radius R_H . Be n' and R'_H the values when no bedform exists. So we have

$$\bar{u} = \frac{1}{n'} R'^{\,3/2}_H S^{1/2} \qquad (8)$$

By combining (7) and (8) we get:

$$\frac{n'}{n} = \left(\frac{R'_H}{R_H}\right)^{3/2} \qquad (9)$$

and by combining (6) and (8) we get :

$$n' = \frac{d_{65}^{1/6}}{24} \qquad (10)$$

Equations (9) and (10) enable to ascertain whether there is a bedform drag or not and to calculate R'_H if need be. This is the case when direct measurement were made of the mean velocities for example at a permanent gauging station.

b. Logarithmic Type Formula

Einstein and Barbarossa chose the following equation which was derived from Nikuradse's experiments by Keulegan.

$$\frac{\bar{u}}{u'_*} = \frac{2.3}{k} \log \left(\frac{12.27\, R'_H\, x}{d_{65}}\right) \qquad (11)$$

where k is the Prandtl - Von Karman coefficient equal to 0.4 for clear fluid and, x , is a correction factor for the transition from hydraulically (see Annex 2 for a discussion about k) rough to hydraulically smooth surface, the roughness being in turn related to the ratio $\frac{d_{65}}{\delta}$, where δ is the thickness of the so-called laminar sublayer and is defined as :

$$\delta = \frac{11.6\,\nu}{u'_*} \quad (\nu : \text{kinematic viscosity of the fluid}) \quad (12)$$

In figure 2, the factor x is given as a function of $\frac{d_{65}}{\delta}$.

Use of Manning-Strickler's formula is recommended when the grain roughness produces a hydraulically rough surface, i.e. when $\frac{d_{65}}{\delta}$ is more than about 5. Whereas use of a logarithmic formula when $\frac{d_{65}}{\delta}$ is less than about 5 (see fig. 2).

In case direct measurements of velocities are made, a trial and error procedure is used to determine R'_H and x. The chosen values have not only to verify equation (11) but to verify both the 2 functions depicted by the curves given in figures 2 and 3.

1.4 Step by Step Procedure for Hydraulic Calculations

Once a test reach has been selected, the following informations are needed.

1. Slope
2. Description of the cross section, that is,

2.1	Curve of R_H versus D	A : Cross section area
2.2	Curve of A versus D	D : Depth or stage
2.3	Curve of P versus D	P : Wetted perimeter

3. Bed sediment distribution curve

The determination of the depth (or stage) - discharge relation proceeds as follows:

1. Select a value of R'_H
2. Calculate u'_* and δ through equations (2) and (12) respectively
3. Determine x from fig. 2
4. Calculate $\bar{u}$ through equation (6) or equation (11)
5. Calculate Ψ_{35} from equation (5)
6. Obtain $\frac{\bar{u}}{u''_*}$ from fig. 3 then calculate u''_* and R''_H
7. Calculate $R_H = R''_H + R'_H$
8. Determine A and D through the description of the cross section
9. Calculate $Q = \bar{u}\ A$

Remark

In flume experiments a side-wall correction is introduced to take into account differences in roughness between the sand-covered bed and the flume walls. In most natural streams such a correction need not be applied.

2. BED-MATERIAL LOAD CALCULATION

The bed-material transport is calculated in its two modes, namely, bed-load and suspended-load for each grain fraction of the bed at each given flow depth.

The procedure used to compute the suspended-load is based on the so-called Rouse equation which is in turn an application of the diffusion-dispersion model.

The Einstein's bedload function is used to calculate the bedload rate. Some theoretical considerations are in place here to shed some light on the procedure.

2.1 Rouse Equation for Vertical Distribution of Suspended Matter

Let us consider particles of uniform shape, size and density in a two dimensional, uniform, turbulent flow.

Since the particle continuously settles with its settling velocity in relation to the surrounding fluid an equilibrium suspension is possible only if the flow provides a countermotion with an equal velocity. This upward movement is due to the turbulence of the flow, which turbulence results from eddies that are being formed continuously and are swirling in an irregular manner as they are carried along by the flow.

The diffusion-dispersion theory states that the settling rate due to gravity per unit area is balanced by the upward movement due to diffusion. This can be expressed by the following equilibrium equation

$$vc = - E_s \frac{dc}{dy} \tag{13}$$

where

v is the settling velocity of the given particle and c the concentration at the height y above the bed. v is given with fig. 4 as a function of the particle diameter, the curve due to Rubey will roughly describe the sediment of most streams.

E_s being a function of y which has been found to be proportional to the diffusion coefficient E_m so we have:

$$E_s = \beta E_m \tag{14}$$

In most applications the β factor is taken as unity. Though experiments have shown that β decreases when both the diameter d and the sediment concentration increase such changes are small in comparison with the changes observed in k.

Furthermore, the local shear stress, that is, the shear stress at the height y above the bottom can be expressed as:

$$\tau_y = \rho E_m \frac{du}{dy} \tag{15}$$

Assuming the Karman-Prandtl velocity law valid, that is,

$$\frac{u - u_{max}}{u_*} = \frac{2.3}{k} \log \frac{y}{D} \tag{16}$$

We finally get the so-called Rouse equation (see Annex 3 for the derivation of this equation).

$$\frac{c}{c_a} = \left[\frac{a (D - y)}{y (D - a)}\right]^{\frac{v}{\beta k u_*}} \tag{17}$$

The quantity $\frac{v}{\beta k u_*}$ is often denoted z. It has been found that the discrepancies observed between theoretical values of z and the ones based on experiments are chiefly due to variations of the k factor. So taking β as unity as well as using for v the settling velocity in clear, still water do not seriously change the z values. (See Annex 2).

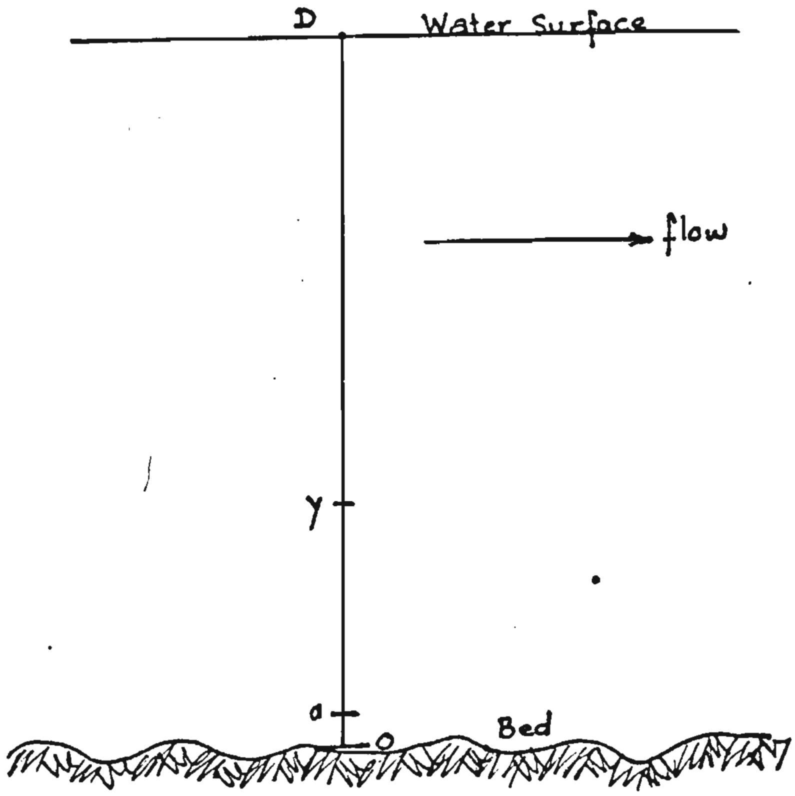


Figure 1

So relation (5) may be used to calculate the concentration, c , of a given grain size whose diameter is, d , at a distance, y , above the bed provided that the concentration, c_a , at a distance, a , above the bed is available. See fig. 1.

2.2 Suspended Load Equation

To obtain the suspended load rate in weight per unit time and unit width, denoted g_{ss}, we have to integrate the product of the velocity and the concentration over the part of the vertical concerned with suspended load, say from a to D.

$$g_{ss} = \int_a^D cu\,dy \tag{18}$$

This time, we use for the velocity distribution the following relation due to Keulegan which relates the velocity not only to the depth y but to d_{65} as well.

$$\frac{u}{u'_*} = \frac{2.3}{k} \log \frac{30.2}{d_{65}} yx \qquad (19)$$

Substituting the Rouse equation (17) for c and equation (19) for u into (18) we get: (see Annex 4 for the derivation)

$$g_{ss} = \frac{2.3}{k} c_a u'_* D \left(\frac{A_E}{1-A_E}\right)^z \left[\log\left(\frac{30.2\ Dx}{d_{65}}\right) \int_{A_E}^{1} \left(\frac{1-y}{y}\right)^z dy + \int_{A_E}^{1} \left(\frac{1-y}{y}\right)^z \log y\ dy \right] \qquad (20)$$

where $A_E = \frac{a}{D}$

According to equation (20) when y approaches zero the concentration becomes infinite, obviously this is not true. In fact the sediment distribution does not apply right at the bed because the concept of suspension, that is, solid particles being continuously surrounded by the fluid fails and so the problem is to determine the thickness of the layer above which suspension is possible and under which takes place the so-called bedload which is actually the source of the suspended load.

2.3 Einstein's Bed-Load Formula

For mixtures with small size spread the total bedload transport of the mixture can be determined directly by using d_{35} as the effective diameter, that is the case when only the bulk rate is needed to predict scour or deposition or when the suspended load is negligeable. The case was dealt with in a previous note entitled "Bedload measurement and sampling."

A few more parameters come up when transport rates of each size fraction have to be computed, mainly to take into account the fact that particles of different sizes in a mixture have not the same behaviour as uniform bed materials.

In that case, the "intensity of bed load transport", Φ_*, and "flow intensity", Ψ_*, are expressed respectively by:

$$\Phi_* = \frac{1}{p} \frac{g_s}{\gamma_s} \left(\frac{\rho}{\rho_s - \rho}\right)^{1/2} \left(\frac{1}{g\, d^3}\right)^{1/2} \qquad (21)$$

p being the fraction of bed material in the given grain size whose representative diameter is d.

$$\Psi_* = \xi\, Y \left[\frac{\log 10.6}{\log \frac{10.6\, Xx}{d_{65}}}\right]^2 \left(\frac{\rho_s - \rho}{\rho}\right) \frac{d}{R'_H\, S} \qquad (22)$$

X is defined as a characteristic grain size of the mixture computed as follows

$$X = 0.77 \frac{d_{65}}{x} \qquad \text{if} \qquad \frac{d_{65}}{x} > 1.80\,\delta \quad \text{or} \qquad (23)$$

$$X = 1.39\,\delta \qquad \text{if} \qquad \frac{d_{65}}{x} < 1.80\,\delta \qquad (23')$$

We recall that δ laminar sublayer is equal to :

$$\delta = \frac{11.6\,\nu}{u'_*}$$

Two correction factors are introduced namely ξ and Y.

ξ or "hiding" factor takes into account the fact that small particles seems to hide between larger ones. Fig. 5 depicts the relation between ξ and the ratio $\frac{d_{65}}{X}$.

Y takes into account changes of the lift coefficient in mixtures with various roughness. Fig. 4 depicts the relation between Y and $\frac{d_{65}}{\delta}$.

Once Ψ_* is determined, we get Φ_* through figure 6 which depicts the Einstein's bedload function, namely,

$$1 - \frac{1}{\sqrt{\pi}} \int_{-1/7\Psi_*(-2)}^{1/7\Psi_*(-2)} e^{-t^2}\, dt = \frac{43.5\,\Phi_*}{1 + 43.5\,\Phi_*} \qquad (24)$$

2.4 Bed-Material Load Equation

For a given vertical, it is logical to think that the summation of the bed-load and the suspended load leads to the determination of the bed-material load. In order to relate the concentration c_a to the bed-load, Einstein introduced the notion of bed-layer whose depth is equal to 2d and stated that suspension is possible only above this layer. Assuming a bed-load movement in the bed layer he derived the reference concentration at 2d from the bed as (see Annex 5 for the derivation)

$$c_{2d} = \frac{1}{11.6} \frac{g_s}{2du'_*} = c_a \quad \text{with} \quad a = 2d \qquad (25)$$

Introducing relation (25) into the suspended load equation (20) we get :

$$g_{ss} = \frac{g_s}{0.216} \left[\ln \frac{30.2\ Dx}{d_{65}} \cdot \frac{A^{z-1}}{(1-A)^z} \int_A^1 \left(\frac{1-y}{y}\right)^z . dy + \frac{A^{z-1}}{(1-A)^z} \int_A^1 \left(\frac{1-y}{y}\right)^z \ln y\ dy \right] \qquad (26)$$

The bed-material load denoted g_{st} is given by :

$$g_{st} = g_s + g_{ss} \qquad (27)$$

Substituting (26) into (27) we obtain :

$$g_{st} = g_s\ (P_E I_1 + I_2 + 1) \qquad (27')$$

where

$$P_E = \ln \frac{30.2\ Dx}{d_{65}} \qquad (28)$$

$$I_1 = 0.216 \frac{A_E^{z-1}}{(1-A_E)^z} \int_{A_E}^1 \left(\frac{1-y}{y}\right)^z dy \qquad (29)$$

$$I_2 = 0.216 \frac{A_E^{z-1}}{(1-A_E)^z} \int_{A_E}^1 \left(\frac{1-y}{y}\right)^z \ln y\ dy \qquad (30)$$

The two integrals are not expressible in closed form in terms of elementary functions.

I_1 and I_2 are graphically depicted in figures 8 and 9 respectively for various A_E and z values.

Equation (27') gives a stream's capacity as to how much bed material load it can transport under uniform and steady flow conditions; washload is not included in Equation (27'). In applying the method for a particular watercourse, Einstein (1950) stresses the following points:

(1) The length of a uniform reach should be such that the slope S may be determined accurately;

(2) the channel geometry, the sediment composition, and all other factors influencing the roughness velue n, such as vegetation, etc., should be uniform, so that an average representative cross section may be selected.

So Einstein's (1950) method of computing the bed-material load is elegant and allows the calculation without measuring either the suspended or the bedload matter.

FIGURES

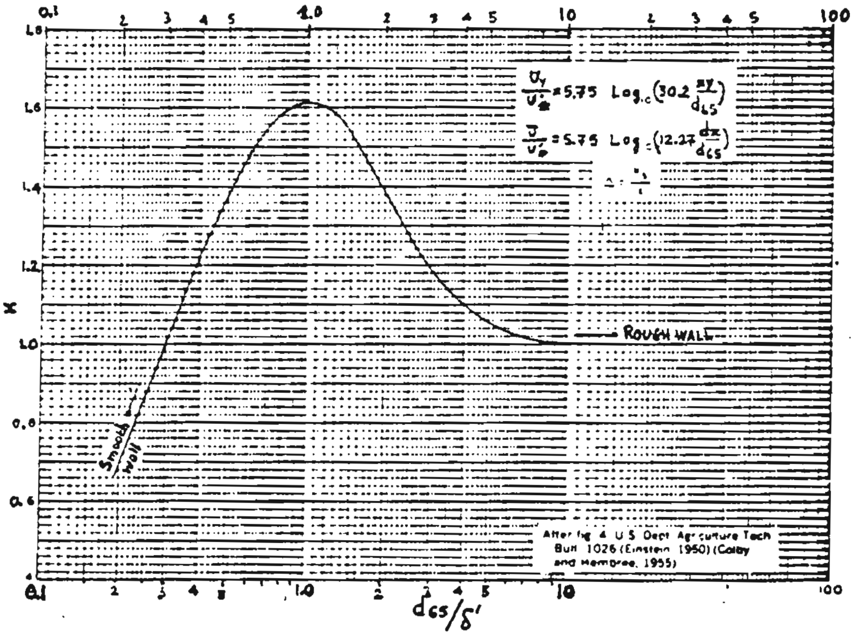


FIG. 2 .—Factor *x* in Velocity Distribution Equation

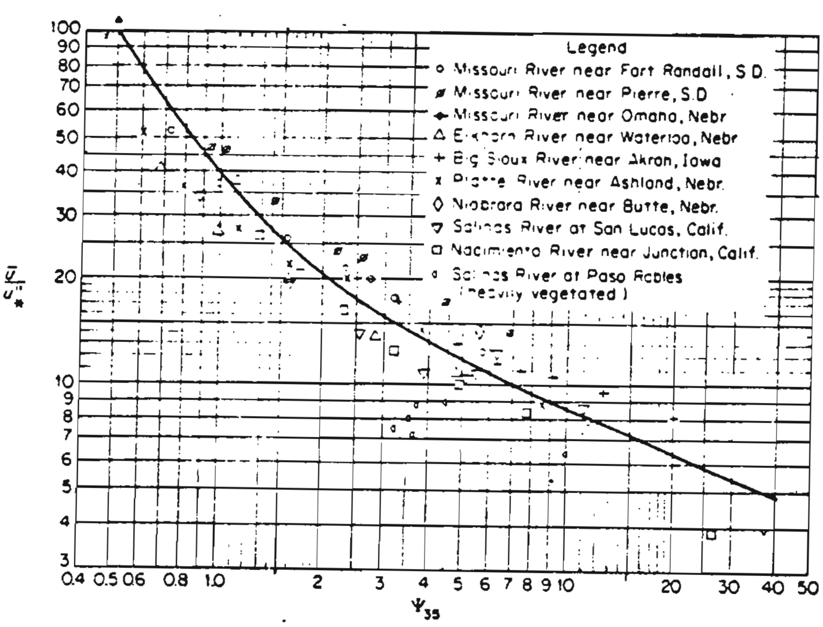


Fig. 3 . Flow resistance due to bedforms. [*After* EINSTEIN *et al.* (*1952*).]

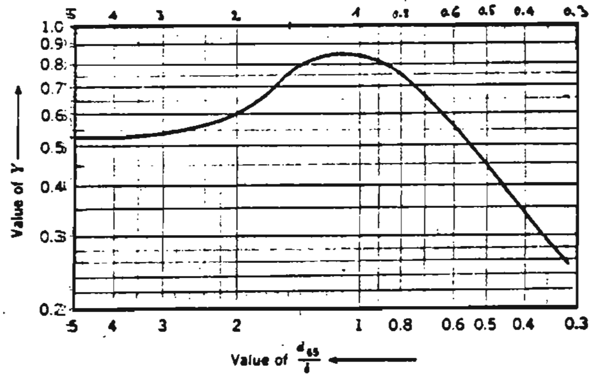


FIG. 4 .—Factor Y in Einstein's Bed Load Function (Einstein, 1950) in Terms of d_{65}/δ

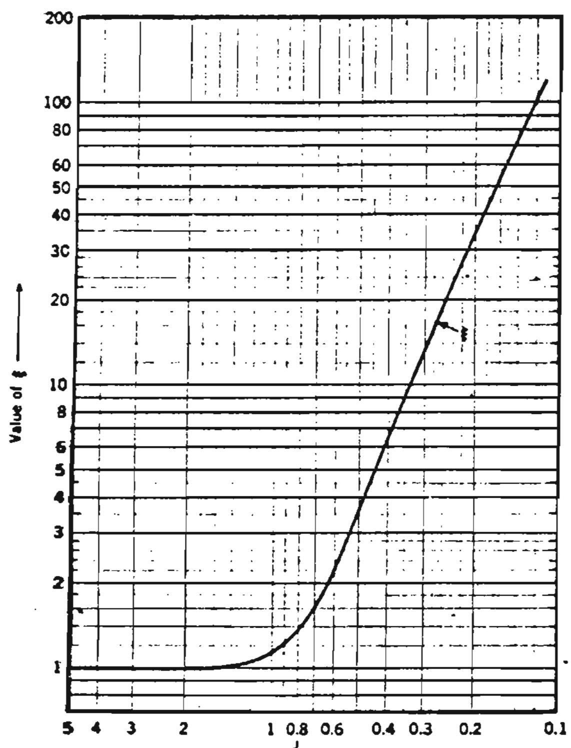


FIG. 5 —Factor ξ in Einstein's Bed Load Function (Einstein, 1950) in Terms of d/X

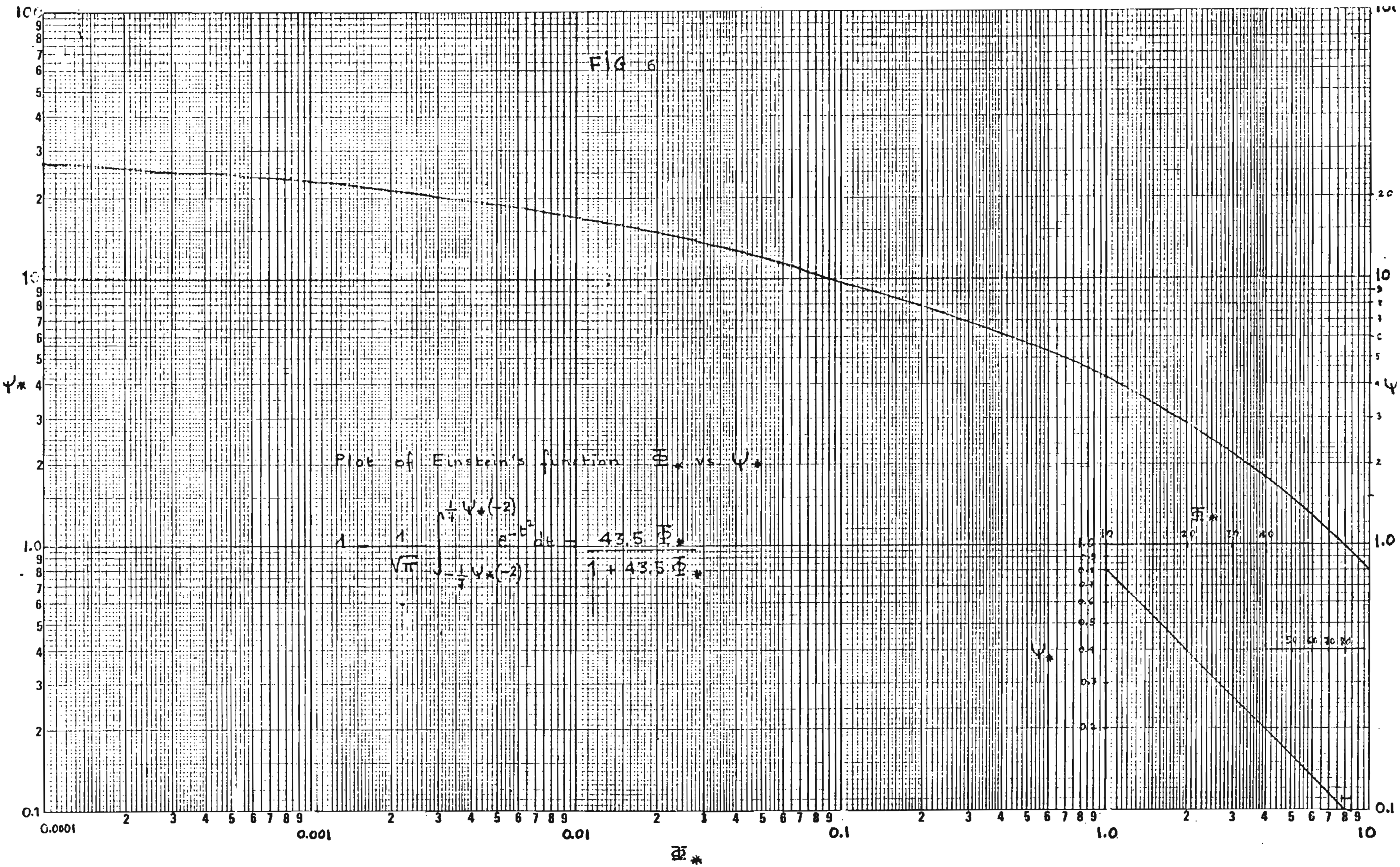


FIG 6
Plot of Einstein's function Φ_* vs. Ψ_*
$1 - \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{7}\Psi_*(-2)}^{\frac{1}{7}\Psi_*(-2)} e^{-t^2}\,dt = \frac{43.5\,\Phi_*}{1 + 43.5\,\Phi_*}$
Ψ_*
Ψ
Φ_*
0.0001
0.001
0.01
0.1
1.0
10

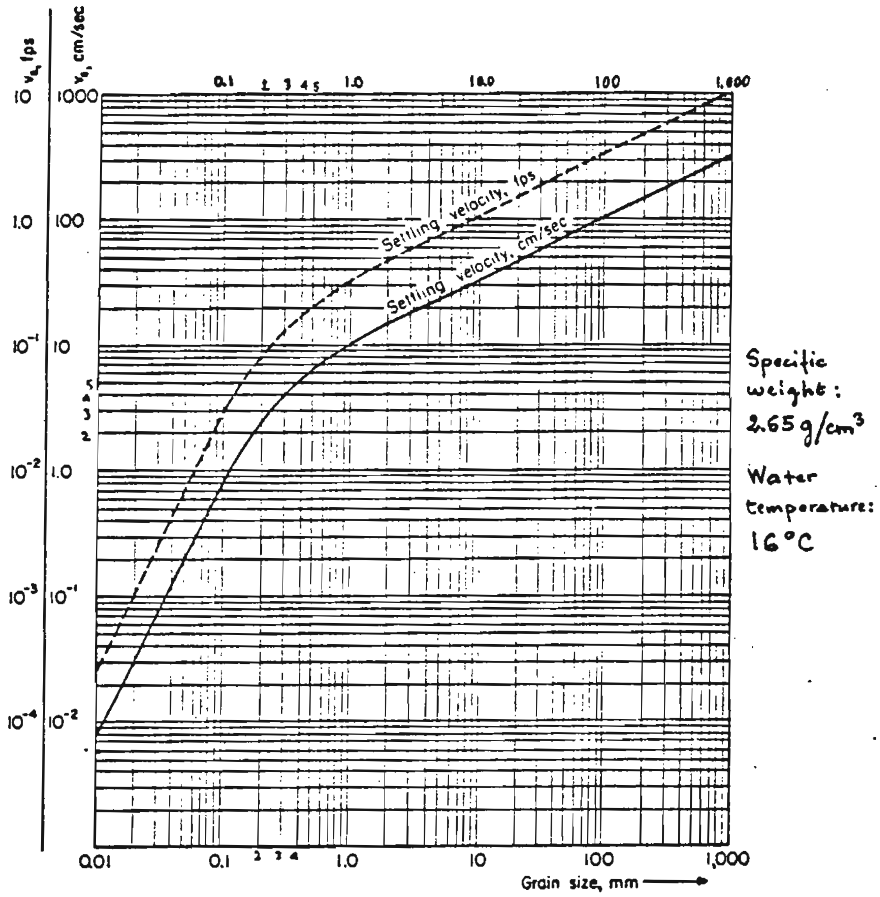


FIG. 7 . Settling velocity v_s for quartz grains of various sizes according to Rubey [10].

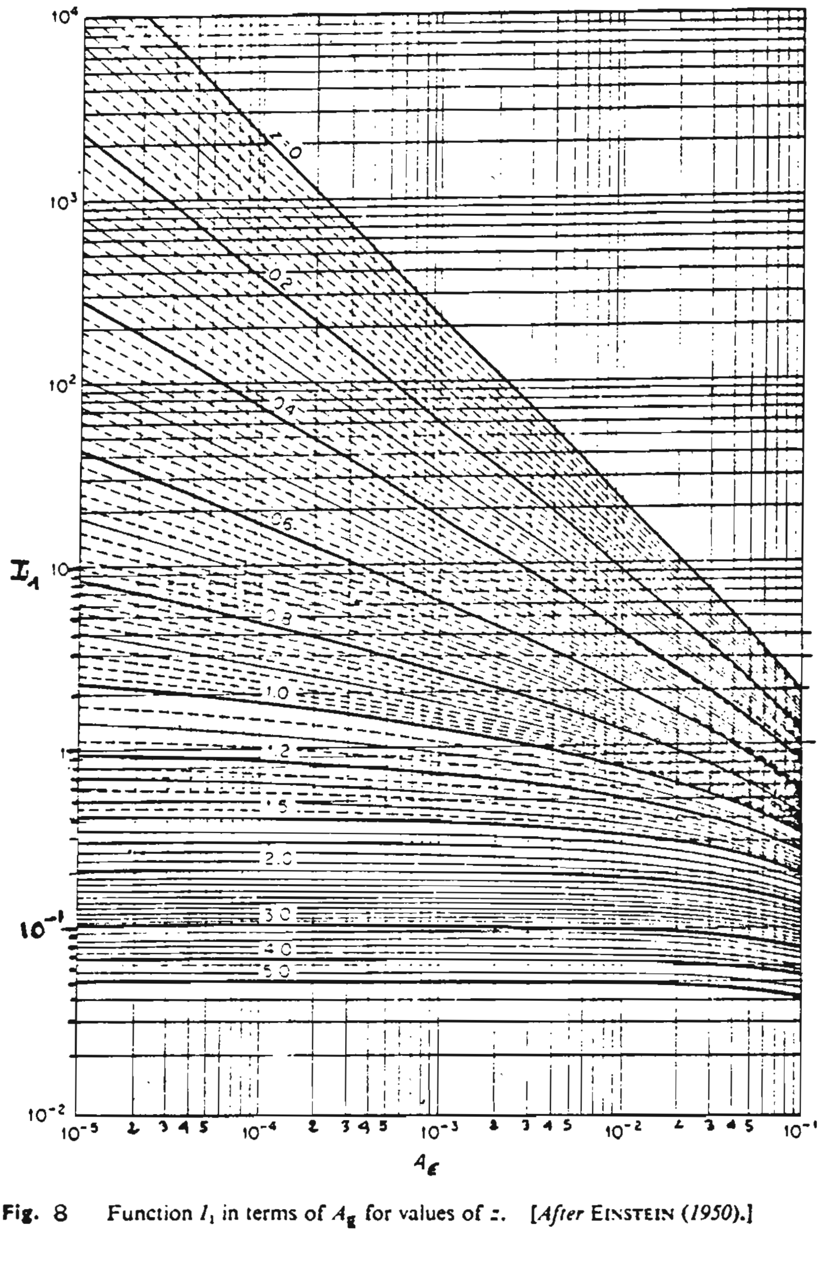


Fig. 8 Function I_1 in terms of A_E for values of z. [*After* EINSTEIN (*1950*).]

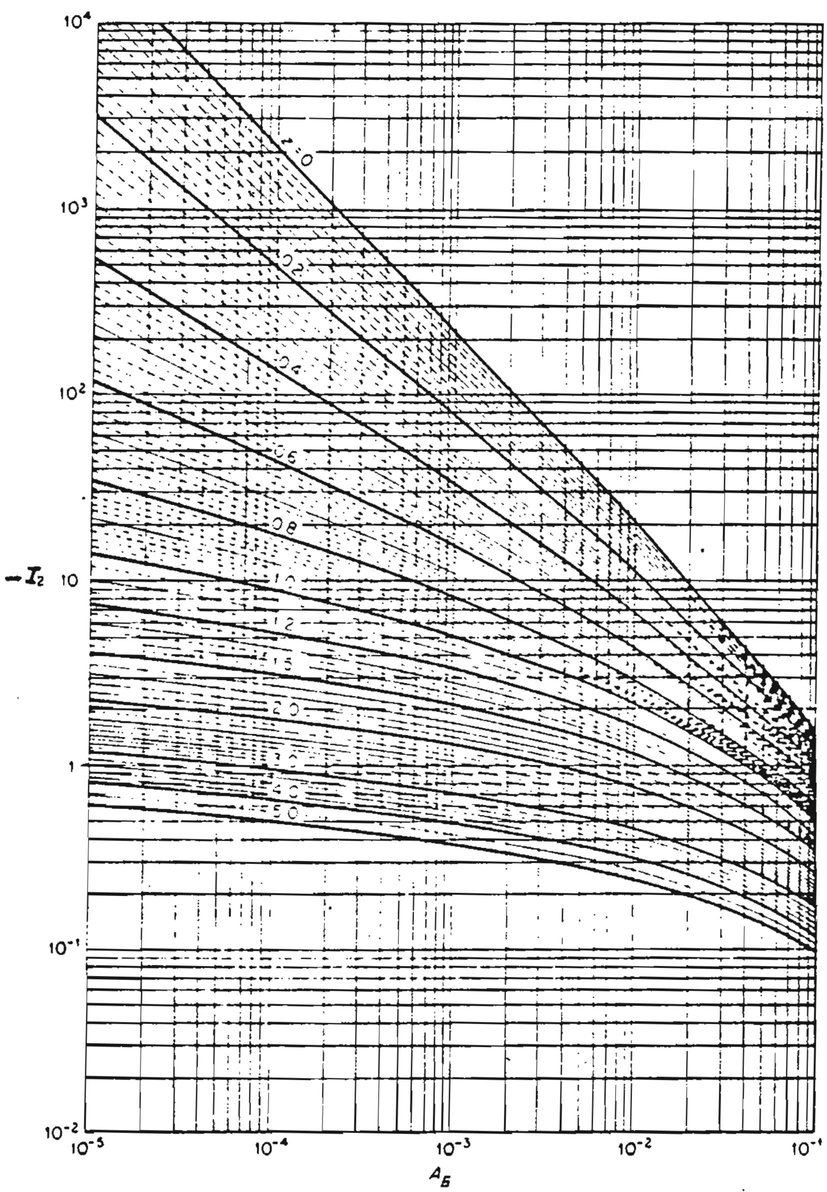


Fig. 9 Function I_2 in terms of A_E for values of z. [*After* Einstein (*1950*).]

(I_2 is negative)

3. EXAMPLE OF BED-MATERIAL LOAD CALCULATION

(After GRAF "Hydraulics of Sediment Transport" p.222)

A test reach, representative of the watercourse to be investigated, has been selected. It was concluded that the channel can be represented by a trapezoidal cross section with bank slopes of 1:1 and a bottom width of 91.45 m. The channel slope was determined and given by $S = 0.0007$.

Five samples, taken down to a depth of approximately 2 ft, were collected to obtain information on the grain size distribution of the entire wetted perimeter. The average values of the five samples are given in table 1.

Table 1

Grain Size Distribution, mm	Average Grain Size	
	mm	Percentage
$d > 0.589$		2.4
$0.589 > d > 0.417$	0.495	17.8
$0.417 > d > 0.295$	0.351	40.2
$0.295 > d > 0.208$	0.248	32.0
$0.208 > d > 0.147$	0.175	5.8
$0.147 > d$		1.8

The average grain size is the geometric mean between the upper and the lower limits of each division, i.e. $0.495 = \sqrt{0.589 \times 0.417}$.

The grain size distribution curve is given in fig. 10.

Description of cross section is given in fig. 11.

Hydraulic calculations are presented in Table 2 and bed material load in table 3. The table heading, its meaning and calculation are explained with footnotes.

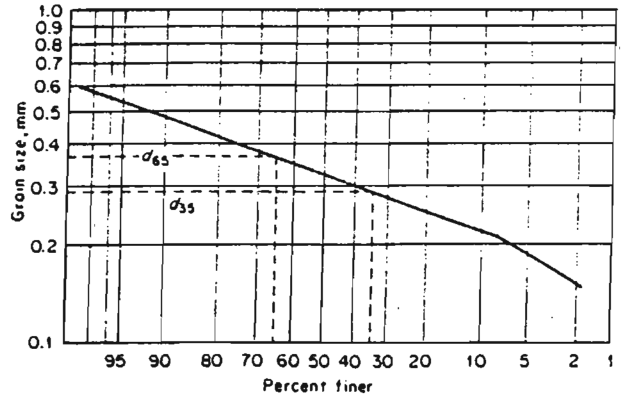


Fig. 10 Grain size distribution of bed material.

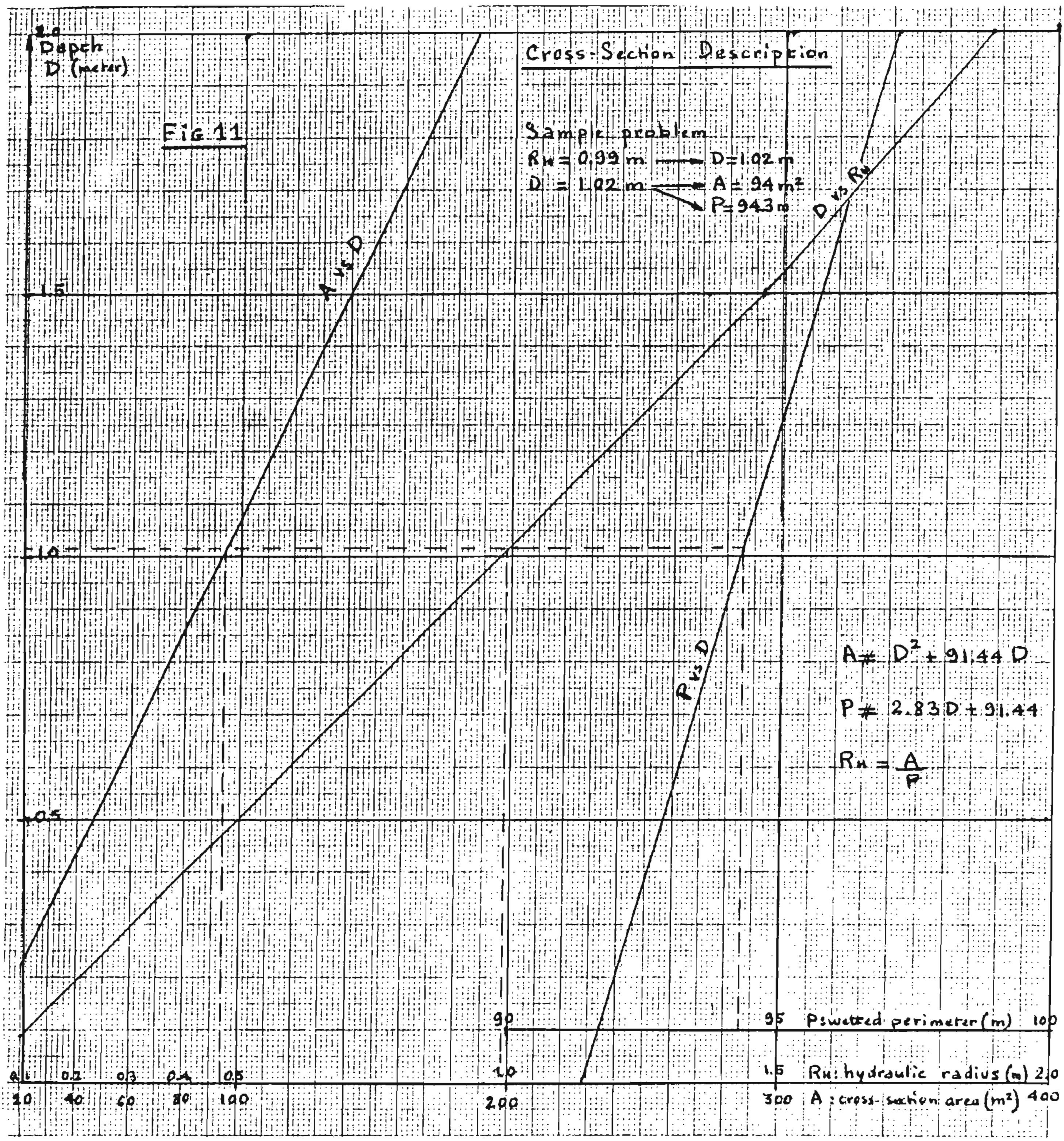


Table 2 Hydraulic calculation for sample problem

R'_H	u'_*	$10^3\delta$	d_{65}/δ	x	$10^3 d_{65}/x$	$\bar{u}$	Ψ_{35}	$\bar{u}/u''_*$	u''_*	R''_H
1	2	3	4	5	6	7	8	9	10	11
0.61 (2 ft) m	0.0647 m/s	0.179 m	1.96	1.40	0.25 m	1.745 m/s	1.12	34	0.51 m/s	0.379 m

(1) Values of R'_H are assumed

(2) $u'_* = \sqrt{gR'_H S}$ friction velocity due to grain roughness

(3) $\delta = \frac{11.6\,\nu}{u'_*}$ laminar sub layer. ν (kinematic viscosity) at 20°C
$\nu = 10^{-2}\text{cm}^2/\text{s} = 10^{-6}\text{m}^2/\text{s}$.

(4) d_{65}/δ

(5) $x = \text{fct}\,(d_{65}/\delta)$ given with fig. 2 Correction factor for roughness transition.

(6) d_{65}/x apparent roughness diameter

(7) $\bar{u} = u'_*\; 5.75 \log \frac{12.27\, R'_H x}{d_{65}}$

(8) $\Psi_{35} = \frac{\rho_s - \rho}{\rho} \cdot \frac{d_{35}}{R'_H S}$ flow intensity with d_{35} as representative diameter

(9) $\frac{\bar{u}}{u''_*} = \text{fct}\,(\Psi_{35})$ given with fig. 3

(10) $u''_* = \left(1/\frac{\bar{u}}{u''_*}\right)\bar{u}$ friction velocity due to bedform drag

(11) $R'' = \frac{u''^2}{gS}$ hydraulic radius due to bedform drag

(12) $R_H = R'_H + R''_H$ hydraulic radius

Table 2 (Continued)

R_H	u_*	D	A	P	Q	$10^3 x$	Y	α	$\left(\frac{\log 10.6}{\alpha}\right)^2$	P_E
12	13	14	15	16	17	18	19	20	21	22
0.99	0.083	1.02	94	94.3	164	0.249	0.60	1.024	1.003	11.72
m	m/s	m	m^2	m	m^3/s	m				

(13) $u_* = \sqrt{gR_H S}$ friction velocity

(14) $D = fct(R_H)$ given with fig. 11 Depth

(15) $A = fct(D)$ given with fig. 11 Cross Section Area

(16) $P = fct(D)$ given with fig. 11 Wetted perimeter

(17) $Q = \bar{u}A$ Water discharge

(18) $X = 0.77 \frac{d_{65}}{x}$ if $\frac{d_{65}}{x} > 1.80$ Characteristic grain size

or $X = 1.39 \delta$ if $\frac{d_{65}}{x} < 1.80$

(19) $Y = fct\ (d_{65}/\delta)$ given with fig. 4 Pressure correction term

(20) $\alpha = \log \frac{10.6\ Xx}{d_{65}}$

(21) $\left[\frac{\log 10.6}{\alpha}\right]^2$

(22) $P_E = 2.303 \log \left(\frac{30.2\ Dx}{d_{65}}\right)$

Table 3 Bed material load calculations for sample problem

R'_H	$10^3 d$	p	d/X	ξ	Ψ_*	Φ_*	g_s	G_s	$\sum G_s$
1	2	3	4	5	6	7	8	9	10
0.61	0.495	0.178	1.99	1.00	1.15	6.7	0.140	13.202	13.202
	0.351	0.402	1.25	1.01	0.82	9.6	0.271	25.555	38.757
	0.248	0.320	1.00	1.13	0.65	12.2	0.160	15.088	54.637
	0.175	0.058	0.70	1.60	0.65	12.2	0.018	1.697	56.334
m	m						kg/m-sec	kg/sec	kg/sec

(1) R'_H

(2) d taken from fig. 10 and Table 1 — grain size diameter

(3) p taken from table 1 — fraction of bed material whose diameter is d

(4) $\frac{d}{X}$

(5) ξ = fct (d/X) given in fig. 5 — hiding factor

(6) $\Psi_* = \xi \, Y \left[\frac{\log 10.6}{\alpha}\right]^2 \left(\frac{\rho_s - \rho}{\rho}\right) \left(\frac{d}{R'_H S}\right)$ — flow intensity on individual grain size

(7) Φ_* = fct (Ψ_*) given in fig. 6 — intensity of transport for individual grain size

(8) $g_s = p \, \Phi_* \, \gamma_s \sqrt{\frac{\rho_s - \rho}{\rho}} \sqrt{g} \sqrt{d^3}$ — bedload rate in weight per unit time and width for a size fraction

(9) G_s = Pgs — bedload rate in weight per unit time for a size fraction for the entire cross-section

(10) $\sum G_s$ — bedload rate in weight per unit time for all size fractions for entire cross-section

So according to Einstein's procedure the bedload rate is in the region of 56 kg/s.

Table 3 (Continued)

$10^3 A_E$	v	z	I_1	$-I_2$	$P_E I_1 + I_2 + 1$	g_{st}	G_{st}	$\sum G_{st}$
11	12	13	14	15	16	17	18	19
0.97	0.063	2.43	0.15	0.95	1.760	0.246	23.198	23.198
0.61	0.045	1.74	0.27	1.80	2.36	0.640	60.352	83.550
0.49	0.035	1.35	0.51	3.00	3.98	0.636	59.975	143.525
0.34	0.022	0.85	2.70	10.0	22.64	0.396	37.362	180.887
	m/s					kg/m-sec	kg/sec	kg/sec

(11) $A_E = \frac{2d}{D}$ ratio of bed layer to water depth

(12) $v = fct(d)$ given with fig. 7 Settling velocity

(13) $z = \frac{v}{0.4\, u'_*}$

(14) $I_1 = f(A_E, z)$ given with fig. 8

(15) $I_2 = f(A_E, z)$ given with fig. 9

(16) $P_E I_1 + I_2 + 1$

(17) $g_{st} = g_s(P_E I_1 + I_2 + 1)$ bed material rate in weight per unit time and width for a size fraction

(18) $G_{st} = P g_{st}$ bed material rate in weight per unit time for a size fraction for the entire cross-section
(P : wetted perimeter)

(19) $\sum G_{st}$ bed material rate in weight per unit time for all size fractions for the entire cross section

Obviously the digits (given by using a calculator) after the decimal point in column 19 are not significant, at best the number of significant figures is 3.

So according to Einstein's procedure the bed material load rate is in the region of 180 kg/s.

CONCLUDING REMARKS

Several items in Einstein's method were questioned. For instance to use u'_* instead of u_* in calculating z in the suspended load equation may seem inappropriate because the diffusion coefficient E_m upon which the equation is based is likely to depend on the total shear stress τ_o and not only on τ'_o, let alone that taking 0.4 for k is also questionable.

Anyway any method has its own limitations and is at best for the time being a mere estimate even though all pertinent variables are taken into account to set it up as it is the case in the Einstein's method.

In the foregoing chapters it was assumed that at any time the sediment bed could afford a continuous and full availability of its particles to be transported under any likely hydraulic conditions, if not, that is, if the supply were partially exhausted the stream would obviously transport less material and a bed material load equation which is supposed to give the maximum capacity (load capacity) would fail.

Last but not least, wherever washload plays an essential role the bed material equations are merely helpful for the understanding of the problem but cannot give correct results since not only such equations are of no help to determine the washload rate but the parameters used to derive them are most likely to undergo drastic changes due to the very presence of the washload (i.e. the factor k which is no longer equals to 0.4 when heavy sediment laden flows are considered).

Annex 1

The following table shows that in the lower regime the values of R''_H are likely to be high as the form roughness predominates whereas in the upper regime when grain roughness predominates R''_H is often negligeable and $R_H \# R'_H$

Classification of bedforms and other information [*after* SIMONS *et al.* (*1965*) *and* SIMONS *et al.* (*1966*)]

Flow regime	*Bedform*	*Bed material concentrations, ppm*	*Mode of sediment transport*	*Type of roughness*	*Roughness, $C/\sqrt{g}$*
Lower regime	Ripples Ripples on dunes Dunes	10–200 100–1,200 200–2,000	Discrete steps	Form roughness predominates	7.8–12.4 — 7.0–13.2
Transition	Washed-out dunes	1,000–3,000		Variable	7.0–20.0
Upper regime	Plane beds Antidunes Chutes and pools	2,000–6,000 2,000 → 2,000 →	Continuous	Grain roughness predominates	16.3–20 10.8–20 9.4–10.7

A useful flow regime criterion is the Froude number denoted N_F and defined as :

$$N_F = \frac{\bar{u}}{\sqrt{g\bar{D}}}$$

where $\bar{u}$ is the stream mean velocity and $\bar{D}$ the mean depth over the entire cross-section.

A rough classification is as follows:

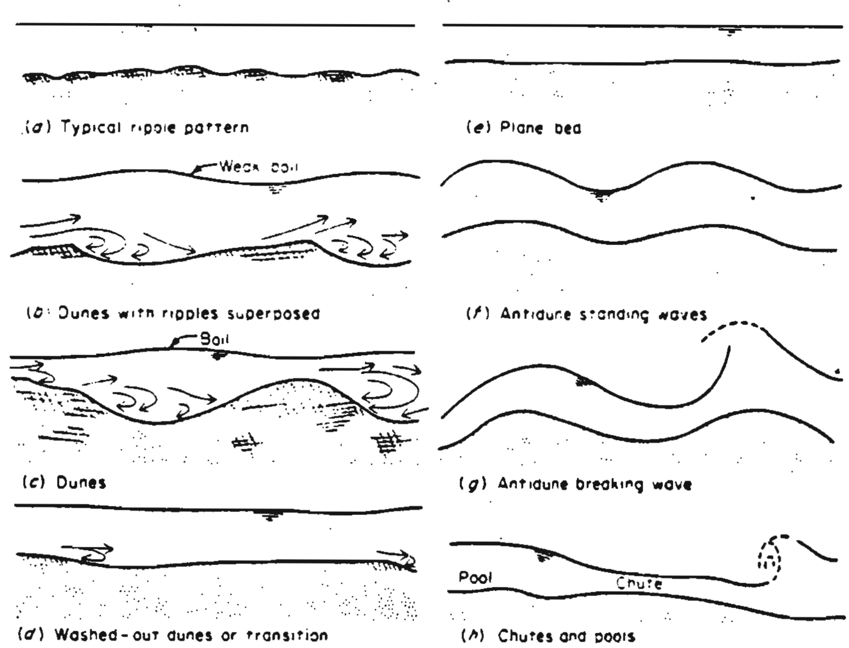
$N_F < 1$ tranquil (streaming) flow - lower regime

$N_F = 1$ critical flow - transition regime

$N_F > 1$ rapid (shooting) flow - upper regime

Annex 1 (Continued)

Sketches of various bedforms are shown in the following figure.



Idealized bedforms in alluvial channels. [*After* SIMONS *et al.* (*1961*).]

It is worth noting that should the bedform change for the same depth (or stage) both the velocity and the water discharge would in turn do, sometimes discontinuous rating curves or rating curves with loops may be interpreted in this way.

To explain the fact that in the upper regime the depth-discharge relation is reasonably stable we will quote Einstein and al.

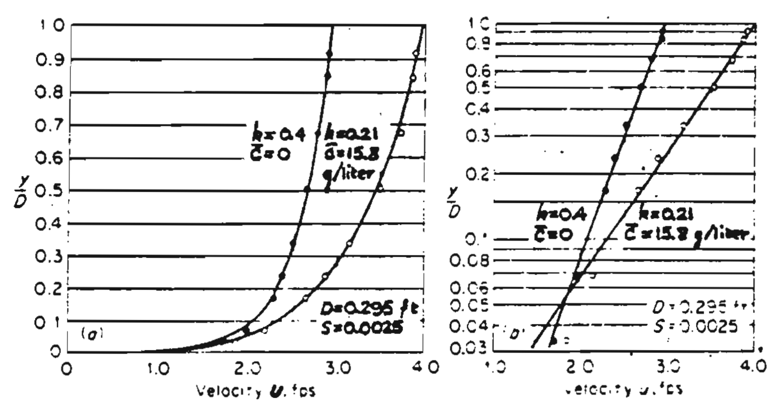
> The effect of irregularities (bedforms) is to distort the flow pattern. When the discharge is least, the distortion of the flow pattern is greatest; as witness the meandering of natural streams at low flows. As the discharge increases and hence the sediment transport along the bed also increases, the distortion of the flow pattern becomes less and less because the alinement of flow becomes progressively straighter. Consequently, one may expect that the additional friction loss, u_*'' , diminishes as the discharge increases.

Annex 2

Variations of k

The value of k is approximately 0.4 for clear fluids, but it has been observed to diminish to as low as 0.2 in flows with high concentration of suspended material. The following figure shows that the logarithmic velocity distribution law holds true but with different values of k according to the mean concentration.

THE SUSPENDED LOAD



Velocity profiles for clear-water and sediment-laden flow. [*After* VANONI *et al.* (1960).]

It has been suggested that a reduction of k means that mixing is less effective and that the presence of sediment suppresses or damps the turbulence.

Anyhow drastic changes may arise in the velocity distribution when high concentrations take place but in that case it is likely that the bulk of the total load is made up of particles finer than the bed material ones and so wash-load is the predominant form of transport.

Annex 3

Derivation of the Rouse Equation

We have the following set of equations

$vc = - E_s \frac{dc}{dy}$ (1) Equilibrium equation

$\tau_y = E_m \frac{du}{dy}$ (2) E_m diffusion coefficient in the diffusion theory

$E_s = \beta E_m$ (3) β constant

$\tau_o = \gamma SD$ (4) Bottom shear stress, often simply called shear stress

$\frac{\tau_y}{\tau_o} = \frac{D-y}{D}$ (5) Ratio of the local shear stress to the bottom shear stress

$u_* = \sqrt{\frac{\tau_o}{\rho}}$ (6) Shear stress velocity or friction velocity

$\frac{u-u_{max}}{u_*} = \frac{2.3}{k} \log \frac{y}{D}$ (7) Karman-Von Prandtl law

Let's take the derivative in equation (7) noting that $2.3 \log \frac{y}{D} = \ln \frac{y}{D}$ we get :

$$\frac{du}{dy} = \frac{u_*}{ky} \qquad (8)$$

Let's express τ_y in terms of τ_o in equation (2) by means of equation (5) we get :

$$\left(\frac{D-y}{D}\right) \tau_o = E_m \rho \frac{du}{dy} \qquad (9)$$

Annex 3 (Continued)

Substituting equation (8) into equation (9) and expressing τ_o in terms of u_* by means of equation (6) we get :

$$\left(\frac{D - y}{D}\right) u_* = \frac{Em}{ky} \qquad (10)$$

Combining equation (3) and (10) E_s can be expressed by :

$$E_s = \beta k u_* y \left(\frac{D - y}{D}\right) \qquad (11)$$

Substituting equation (11) into equation (1) and separating the variables we get :

$$\frac{dc}{c} = - \frac{v}{\beta ku_*} \frac{Ddy}{y(D - y)} \qquad (12)$$

Let us assume that the concentration of suspended sediment at a point a is c_a. Then integrating (12) from a to y we get :

$$\ln \frac{c}{c_a} = \frac{-v}{\beta ku_*} \int_a^y \frac{Ddy}{y(D - y)} = \frac{v}{\beta ku_*} \left[-\ln\left(\frac{y}{D-y}\right)\right]_a^y = \frac{v}{\beta ku_*} \log \frac{a(D-y)}{y(D-a)}$$

and taking the antilogarithms

$$\frac{c}{c_a} = \left[\frac{a(D-y)}{y(D-a)}\right]^{\frac{v}{\beta ku_*}}$$

The quantity $\frac{v}{\beta ku_*}$ is often called z.

Annex 4

Derivation of the Suspended Load Equation

We have the following three relations :

$$\frac{u_y}{u_*} = 5.75 \log \frac{30.2\,xy}{d_{65}} = \frac{1}{0.4} \ln \frac{30.2\,xy}{d_{65}} \qquad (1)$$

$$\frac{c_y}{c_a} = \left[\frac{a(D-y)}{y(D-a)}\right]^{z} \qquad (2)$$

$$g_{ss} = \int_a^D c_y u_y \, dy \qquad (3)$$

Substituting (1) and (2) into (3) we get :

$$g_{ss} = 5.75\, u_*\, c_a \left[\log \frac{30.2\,x}{d_{65}} \int_a^D \left[\frac{a(D-y)}{y(D-a)}\right]^{z} dy + \int_a^D \left[\frac{a(D-y)}{y(D-a)}\right]^{z} \log y \, dy\right] \qquad (4)$$

Let us introduce $A_E = \frac{a}{D}$ then we have :

$$\left[\frac{a(D-y)}{y(D-a)}\right]^{z} = \left(\frac{A_E}{1-A_E}\right)^{z} \cdot \left(\frac{1-\frac{y}{D}}{\frac{y}{D}}\right)^{z} \qquad (5)$$

Let us take as new variable $u = \frac{y}{D}$ then we have :

$$dy = D\,du$$

and the new limits of integration are $u = A_E$ and $u = 1$ for $y = a$ and $y = D$ respectively.

Annex 4 (Continued)

Consequently we get :

$$\int_a^D \left[\frac{a(D-y)}{y(D-a)}\right]^z dy = D\left(\frac{A_E}{1-A_E}\right)^z \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z du \quad \text{and} \qquad (6)$$

$$\int_a^D \left[\frac{a(D-y)}{y(D-a)}\right]^z \log y \, dy = D\left(\frac{A_E}{1-A_E}\right)^z \left[\int_{A_E}^1 \left(\frac{1-u}{u}\right)^z \log u \, du + \log D \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z du\right] \qquad (7)$$

Substituting (6) and (7) into (4) we finally get:

$$g_{ss} = 5.75\, u_* \, D \, c_a \left(\frac{A_E}{1-A_E}\right)^z \left[\log \frac{30.2\, Dx}{d_{65}} \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z du + \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z \log u \, du\right] \qquad (8)$$

or taking the Naperian logarithms

$$g_{ss} = \frac{1}{0.4}\, u_* \, D \, c_a \left(\frac{A_E}{1-A_E}\right)^z \left[\ln \frac{30.2\, Dx}{d_{65}} \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z du + \int_{A_E}^1 \left(\frac{1-u}{u}\right)^z \ln u \, du\right] \qquad (8')$$

Annex 5

Derivation of the Bed-Material Load Equation

Einstein found that in the so-called laminar sub-layer whose depth is

$$\delta = \frac{11.6\,\nu}{u_*}$$

the bottom velocity, u_B , is related to the shear stress velocity by :

$$U_B = 11.6\, u_*$$

So assuming that the particles in the sublayer move with an average velocity equal to U_B, the bed-load per unit width g_s may be considered as the product of the concentration c_a and the discharge per unit width, so we can write :

$$g_s = c_a\, a\, u_B \qquad \text{or}$$

$$g_s = c_a\, a\, 11.6\, u_*$$

and with $a = 2\,d$ we get :

$$c_a = \frac{g_s}{11.6\, u_*\, 2d} \qquad (1)$$

Let us resume the suspended load equation [Annex 4, equation (8')]

$$g_{ss} = \frac{1}{0.4}\, u_*\, D\, c_a \left(\frac{A_E}{1-A_E}\right)^z \left[\ln\left(\frac{30.2\, Dx}{d_{65}}\right)\int_{A_E}^{1}\left(\frac{1-y}{y}\right)^z dy + \int_{A_E}^{1}\left(\frac{1-y}{y}\right)^z \ln y\, dy\right] \qquad (2)$$

which may be rewritten as follows:

$$g_{ss} = \frac{1}{0.4}\, u_*\, D\, c_a\, A_E \left[\ln\left(\frac{30.2\, Dx}{d_{65}}\right)\frac{A_E^{\,z-1}}{(1-A_E)^z}\int_{A_E}^{1}\left(\frac{1-y}{y}\right)^z dy + \frac{A_E^{\,z-1}}{(1-A_E)^z}\int_{A_E}^{1}\left(\frac{1-y}{y}\right)^z \ln y\, dy\right] \qquad (3)$$

Annex 5 (Continued)

Substituting (1) into (3) and noting that $a = 2d$ and consequently $A_E = \frac{a}{D} = \frac{2d}{D}$ we get :

$$g_{ss} = \frac{1}{0.4} u_* D \frac{1}{11.6} \frac{gs}{2du_*} \frac{2d}{D} \left[. \ . \ . \ . \right] = \frac{gs}{0.216} \left[. \ . \ . \ . \ . \ . \right] \quad (4)$$

Finally we get for the bed-material load

$$g_{st} = g_s + g_{ss} = g_s (P_E I_1 + I_2 + 1)$$

where

$$P_E = \ln \left(\frac{30.2 \ Dx}{d_{65}}\right)$$

$$I_1 = 0.216 \frac{A_E^{z-1}}{(1-A_E)^z} \int_{A_E}^{1} \left(\frac{1-y}{y}\right)^z dy$$

$$I_2 = o.216 \frac{A_E^{z-1}}{(1-A_E)^z} \int_{A_E}^{1} \left(\frac{1-y}{y}\right)^z \ln y \, dy$$

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