Instability Conditions and Energetics in the Equatorial Pacific

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ABSTRACT

Instability conditions applicable to zonal flows in the equatorial oceans are presented. A multi-layer model, with either rigid bottom (including the effects of topography) or "reduced gravity" lower boundary, is employed. Preliminary use of these theoretical results is illustrated with the zonal geostrophic velocity profiles calculated from the Hawaii-Tahiti Shuttle Data (Wyrtki and Kilonsky, 1984). Sufficient stability conditions - or necessary instability ones - come in two different sets : the second one is related to the sign of the energy of growing perturbations.

1. Introduction

In a 11/2-layer model, if there exists a value α such that

$$[U(y)-\alpha]\,(dQ/dy) < 0 \qquad (1a)$$

and,
$$g'H(y) > [U(y)-\alpha]^2 \qquad (2a)$$

for all y, then the zonal flow (u, v, h)≡(U, 0, H) is stable (Ripa 1983)[1]. There are sufficient stability conditions; necessary instability conditions are, therefore, that for any value of α, either (1a) or (2a) or both must be violated.

In a model with two layers (Ripa, 1987, 1989a-b; McPhaden and Ripa, 1990), stability condition (1a) is simply replaced by

$$[U_1-\alpha]\,(dQ_1/dy) < 0 \quad \text{and} \quad [U_2-\alpha]\,(dQ_2/dy) < 0, \qquad (1b\text{-}c)$$

whereas the second condition, (2a), changes into

$$g' > [U_1-\alpha]^2/H_1 + [U_2-\alpha]^2/H_2 \qquad (2b)$$

if the system has a rigid bottom (which might include topography), or into

$$g'^2_2 < [g'_1+g'_2-(U_1-\alpha)^2/H_1]\,[g'_2-(U_2-\alpha)^2/H_2] \qquad (2c)$$

in the "reduced gravity" case (i.e., when the lower boundary is the interface with a third, motionless layer); these last two systems are called the "2-layer" and "21/2-layer" model, respectively.

Two peculiarities of the stability conditions might call the attention :

Firstly, an arbitrary constant α may be subtracted from the fields $U_j(y)$: the variations of U is what matters. Thus uniform flow, U_j = constant is stable to finite amplitude perturbations. In particular, there is no "baroclinic instability" in a 11/2-layer

1. The notation is standard, e.g., g' is the reduced gravity and Q = (f - dU/dy)/H, denotes potential vorticity. The basic flow is in geostrophic balance : fU +g'dH/dy = 0.

model. This powerful property follows from zonal homogeneity of the system equations and boundary conditions (e.g., -plane or the sphere, with topography function only of latitude, and with coasts along parallels), as well as of the basic flow under consideration.

Secondly, there are two different stability conditions, or two sets of them. Even though this is by no means novel, it is because of the absence of the second condition in the quasi-geostrophic models (not applicable to the equatorial zone) that some readers find it difficult to accept. The method used to find the stability conditions, and the relationship of the second one with the sign of perturbation energy is discussed next.

2. Stability conditions and integrals of motion.

The procedure devised by Arnol'd (1965, 1966) consists in finding conditions such that if satisfied by a particular basic flow, [u,h] = [U,H], then a certain integral of motion, S[u,h], has an extreme at the basic solution, i.e.,

$$S[u,h] = \text{constant},$$

for any initial condition, and

$$\delta S = S[U+u', H+h'] - S[U,H] > 0,$$

for all finite perturbations, [u',h'], which are different from zero and sufficiently small. Now, δS is an exact, finite-amplitude, constant of motion (N.B., S[U,H] = constant, because the basic flow is steady) which, to lowest order, is quadratic in the perturbation. This lowest order is composed of two parts : the first one involves the square of the perturbation potential vorticity, and the second one is the perturbation energy[2]. The latter is *not* necessarily positive, because there might exist perturbations that diminish the total kinetic energy more than they increase the total potential energy, by means of anti-correlated changes of velocity and layer thickness. The two stability conditions, e.g. (1a, b, or c) and (2a, b, or c), are precisely those that guarantee that both quadratic parts of δS are positive definite.

Since δS is constant, in the case of an unstable basic flow a small perturbation must have a *vanishing* net value of the quadratic part of δS (Just think of an infinitesimal wave whose amplitude grows exponentially with time.) If the first condition is not violated (for instance, when the basic flow has uniform potential vorticity), then the perturbation energy must be negative (e.g., Marinone and Ripa, 1982) or zero (e.g., Hayashi and Young, 1987).

Non-positive wave energy is present in a well-known phenomenon, namely, Kelvin-Helmholtz instability (e.g., see Miles 1980). However this seems to be ignored by many researchers, particularly those with prejudices formed in the realm of the quasi-geostrophic models, for which the energy of a growing perturbation is always positive (that is why, incidentally, only the first condition is needed in the quasi-geostrophic case). In Kelvin-Helmholtz instability theory, there are no gradients of potential vorticity involved and the energy of a growing perturbation must be exactly zero, whereas the energy of a neutral wave can have either sign.

2. More precisely, "energy - α (zonal momentum)", but I call it "energy", for short.

3. Stability conditions for the N-layer model.

Consider a general N-layer model. A value of N = n1/2-layers with integer n, actually means a (n+1) - layer model in which the deepest layer is at rest; it is, then, a n-layer, system with a "soft" bottom. On the other hand, in a regular N-layer model, i.e. with integer N, the deepest layer has a rigid bottom, which may include topography. I am working with the primitive equations, and making use of the hydrostatic, Boussinesq and "rigid lid" approximations; one can easily deal without the last two, but that only complicates the algebra without much gain in physical insight. The model is completely specified by the total volume of each layer (or the mean thickness, in the case of an unbounded horizonal domain) and the buoyancy jumps across the interfaces, g'_j. The dynamical variables are the thickness, h_j , and horizontal current in each layer u_j; the latter is independent of the vertical coordinate.

The stability of a general basic zonal flow, ($U_j = U_j(y)$, $V_j \equiv 0$, $H_j = H_j(y)$ is guaranteed if there exists a value α such that[3]:

$$[U_j - \alpha]\,(dQ_j/dy) < 0 \qquad (1d)$$

and

$$\mu_j > [U_j - \alpha]^2/H_j \qquad (2d)$$

where the fields μ_j are defined in the following paragraph. These sufficient stability conditions are the generalization of (1a) and (2a) for the multi-layer case. Necessary instability conditions are that, *for any* α, either (1d) or (2d) or both must be violated.

First define $\gamma_j \equiv (U_j-\alpha)^2 / H_j$. For the *reduced gravity* case, n1/2-layer model, first make $\mu_n \equiv g'_n$ and then $\mu_{j-1} \equiv g'_{j-1} - \mu_j\gamma_j / (\mu_j-\gamma_j)$ for j decreasing from n-2 down to 2. For the *rigid bottom* case, n-layer model one starts from $\mu_{n-1} \equiv g'_{n-1} - \gamma_n$ and then one does a similar thing, decreasing j from n-3 down to 2. As a check, notice that one goes from N = n+1 to N = n1/2 by making $H_{n+1} \to \infty$, which implies $\gamma_{n+1} \to 0$. Similarly, one formally goes from N = n1/2 to N = n by making $g'_n \to \infty$; I say "formally" because the n-layer model might have topography, and therefore is not truly derivable from the n1/2-layer one.

Let us review the second condition, for systems with few layers. Perturbation energy must be positive definite if

N = 1:	always	
N = 11/2:	$g' > \gamma$	
N = 2:	$g' > \gamma_1 + \gamma_2$	
N = 21/2:	$g'^2_2 < (g'_1 + g'_2 - \gamma_1)\,(g'_2 - \gamma_2)$	
N = 3:	$(g'_2 - \gamma_3)^2 < (g'_1 + g'_2 - \gamma_3 - \gamma_1)\,(g'_2 - \gamma_3 - \gamma_2)$	
N = 31/2:	$\mu^2 < [g'_1+\mu-\gamma_1]\,[\mu-\gamma_2]$	$\mu=g'_2-\gamma_3 g'_3/(g'_3-\gamma_3)$

where $\gamma_j \equiv [U_j-\alpha]^2/H_j$.

3. In all three cases, the first condition, (1d), must be satisfied in every layer (j = 1, ..., n), whereas the second one, (2d), must be fulfilled for (j = 1, ..., n) in the reduced gravity case or (j = 1, ..., n-1) in the rigid bottom case.

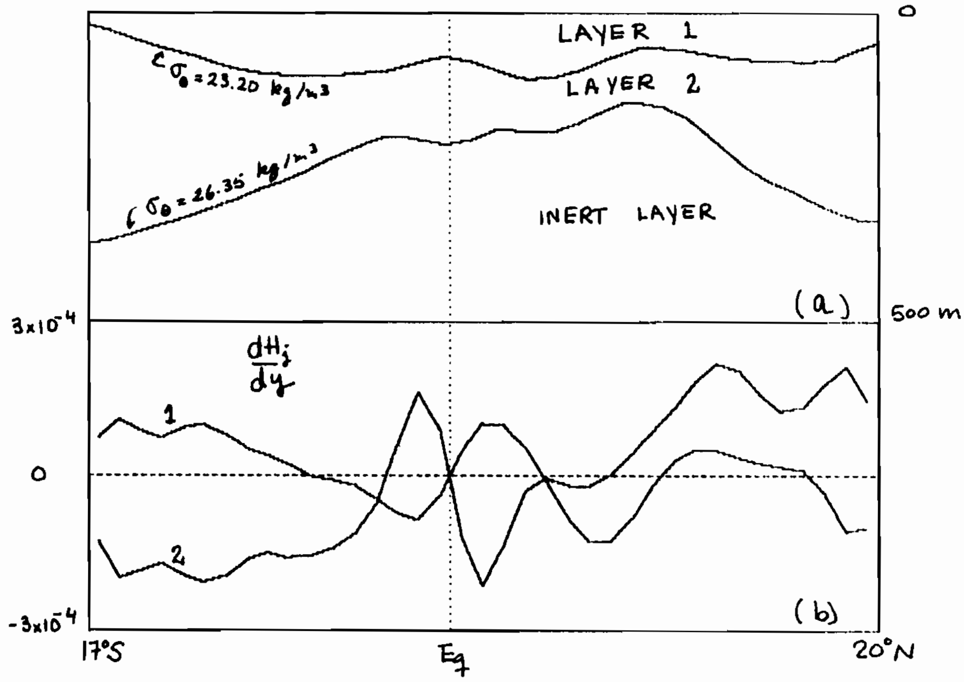

FIG.1. (a) 2 1/2-layer model of the equatorial Pacific, defined by the surface and the σ_θ=23.20 kg.m^{-3} and σ_θ=26.35 kg.m^{-3} isopycnals (the third layer is assumed infinitely deep and motionless). The isopycnals mean depths were calculated from the data of Wyrtki and Kilonsky (1984). (b) Meridional derivatives of the layer thicknesses.

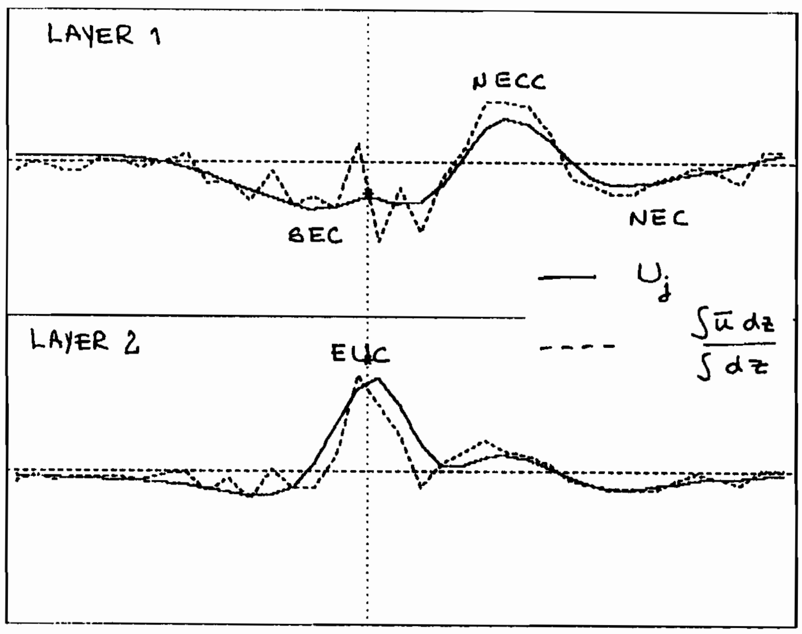

FIG.2. Geostrophic currents in each layer (solid lines) and average velocities calculated from Wyrtki and Kilonsky (1984) continuous field (dashed lines).

4. From one to infinity : the ultraviolet problem.

The second condition (perturbation energy positive definite) is more restrictive on the shear of the basic flow as the number of layers is increased. Moreover, assume that one builds successive approximations to a continuously stratified medium by N-layer models, with increasing N: As the layer thicknesses are diminished, say $H \rightarrow \varepsilon H$, with $\varepsilon < 1$, so are the buoyancy jumps, $g' \rightarrow \varepsilon g'$, but the fields γ are increased, $\gamma \rightarrow \gamma/\varepsilon$, and thus the second condition is harder to satisfy, by a factor of $1/\varepsilon^2$. The inevitable conclusion is that there are no conditions on the basic flow that make the energy of an arbitrary perturbation positive definite : one has to resort to conditions that also involve properties of the disturbance.

For instance, following Holm and Long (1988), one can use as the "second condition" in a continum system.

$$N^2/m^2 > (U-\alpha)^2, \qquad (3)$$

where N is the Brunt-Vaisala frequency (N^2 is the vertical gradient of the basic buoyancy profile) and m is a local vertical wavenumber of the perturbation[4]. Unlike conditions (2a-d), which only involve the basic flow, (3) also bounds the vertical scale of the perturbations : short enough disturbances will violate it (as happenes with the horizontal scale in the case of Kelvin-Helmholtz instability). Therefore, (3) is indeed a condition for linear stability : in a fully nonlinear problem, there is no way to assure that m^2 will be bounded at all times, even if it so initially.

5. Preliminary results of an application to the equatorial Pacific

A model of equatorial currents with some pretense of realism must, at least, have two layers. Prof. Klaus Wyrtki has been kind enough to provide me with the mean annual data shown in the Wyrtki and Kilonsky (1984) paper; figure 1a shows a 21/2-layer model (i.e., the third one is assumed infinitely deep and motionless) with interfaces defined as the σ_θ=23.20 kg.m^{-3} and σ_θ=26.35 kg.m^{-3} isopycnals, respectively. The meridional derivatives of the layer thicknesses, figure 1b, vanish right at the equator, which is a necessary condition for geostrophic balance.

Geostrophic currents in each layer (solid lines in figure 2) show, not surprisingly, good agreement with the average velocities (dashed lines) calculated from Wyrtki and Kilonsky "continuous " field, using buoyancy jumps equal to g'_1 = 27 mm.s^{-2} and g'_2 = 15 mm.s^{-2}. The south equatorial current (SEC), north equatorial countercurrent (NECC) and current (NEC) are clearly seen in the first layer. The equatorial undercurrent (EUC) is observed in the second layer, along with the SEC, NECC (or the subsurface one), and NEC.

The meridional gradient of potential vorticity is always positive everywhere in the first layer and only south of 10°N in the second layer. The first stability condition is violated at the site of NECC, in the first layer, and of the EUC, in the second one, if a value α = 0 is used. However, with α = 60 cm.s^{-1}, the first condition is not violated in those regions, but rather, north of 10°N in the second layer. The second stability condition is found to be satisfied for both values of α.

In summary, the second condition is satisfied (i.e., the vertical resolution is not rich enough to allow for negative energy perturbations) but the first one is violated; the place where this occurs depends on the value of α. A normal mode calculation is

4. The generalization of the "first condition" to the continuum case is straightforward.

underway. I expect to find the flow to be unstable, much like in the work of Philander (1976, 1978), and it will be interesting to compare the localization of a growing perturbation and the regions where (1c) is violated, depending on the value of α.

6. Finale

Laplace tidal equations are probably the archetype of ocean dynamical models. When they are linearized in the deviation from a resting ocean, *two* types of waves are found : Poincaré and Rossby ones (these become gravity waves and vertical modes, respectively, in the absence of Coriolis effects). Two is also the number of conditions that guarantee the stability of the steady solutions of those equations, as discussed here and elsewhere.

It is my conjecture that growing perturbations from an unstable flow that violates (1) but satisfies (2) are Rossby-like. Conversely, an unstable steady solution that fulfills (1) but not (2), is presumed to have growing perturbations which are Poincaré-like. Marinone and Ripa (1982) studied unstable easterly equatorial jets in a 11/2-layer model, finding that the narrower jet violated (1a) but not (2a); the opposite was true for one with a width equal to the deformation radius. The structure of the perturbations in each case was like as described above (Rossby and Poincaré-like, respectively). Of course, I expect my conjecture to hold true not only for the 11/2-layer model, but also for cases with richer vertical resolution, for which the second condition is harder to satisfy.

In the strictly two-dimensional (non-divergent) case as well as in the quasi-geostrophic models, there is only one type of linear waves (i.e., Rossby ones), and also there is only one stability condition, namely that corresponding to (1), in accordance with that expressed above. For all other cases two sets of stability conditions are found. The second condition is associated to the possibility of perturbations with negative or vanishing energy. It is important to examine this concept further, particularly in the equatorial region, where Poincaré waves cannot be avoided in any truly non-linear model.

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