WHY ARE CLOSED SIGN SYSTEMS ISOMORPHIC TO MATHEMATICAL GROUPS?

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I thank Dr Bertrand Gerard for the opportunity to contribute this paper. Parts of the argument have been taken from Lucich(1987) which provides analyses and evidence in greater elaboration

ABSTRACT

Two Australian Aboriginal kinship systems are used as examples of closed sign systems which are isomorphic to one set of abstract groups. The isomorphism allows the pursuit and utilisation of other formal properties such as homomorphism, and the correspondences point to common sets of structures which stand for combined cognitive operations or transitions. The latter in turn are required by practices whiche merge under particular social preconditions, and according to interests that bear homogeneity, reciprocity, alignment and systematisation.

Socially constructed sign systems with properties of closure are found in certain forms of kinship reckoning, in mosaic designs, in cycles of time and number, and in equally tempered musical scales. Their underlying codes or metalanguages are isomorphic to the group-theoretic structures of mathematics. The recovery of multi-generator free groups paves the way for applying the findings of neurobiology as indicatedby Churchland (1986: 220), and the insights of cognitive psychology as suggested in different ways by Cassirer(1944), Piaget(1971), de Mey(1982), and Leyton(1986:125).

The first part of this paper is a demonstration of isomorphism, while the second part points to the bridging assumptions and the societal contexts within which the isomorphisms become relevant. The form of social organization most germane to the demonstration is in the field of elementary kinship, notably that of the Aboriginal Australians. The relevance to kinship cognition is in the distributions of terms and genealogical kintypes onto the models frequently proposed by ethnographers, models characterised by periodicity and symmetry. Such models are also constructed by mathematicians in a tradition that includes Weil(1949/70), White(1963), Courrège(1965), Boyd(1969), Cargal(1978), de Meur and Jorion(1981), and Tjon Sie Fat(1981).

When specific sibling and spouse equation rules formally define equivalences in a genealogieal tree they also entail homogeneous structures which match the axioms of group theory. Those axioms and structures identify patterns, permutations and the limits to the possible. In particular, the mathematical properties of defining relations, isomorphism, conjugacy and homomorphism are paralleled respectively in the spouse equations, deep structures, skew rules and sociocentric classes of the ethnographers' kinship models.

A first step in model construction is to start from the concept of a free group generated by X and Y. The realisation for Australian kinship requires that X and Y each stand for a distinct type of reckoning or tracing action. Generator X stands for the action of tracing the relation or transition to ZHsib (sister's husband's sibling)[1] , and generator Y stands for the action of tracing the relation or transition to BC (brother's child). The actions are regarded as reversible, and the class of ego and siblings is

[1] The convention here is that B = Brother, D = Daughter, F = Father, H =,Husband,M =Mother, S =Son,W=Wife,Z=Sister, and sib= sibling.

This is the presentation ofa group in terms of a free group, since any group is isomoiphic to a factorgroup of a free group (Scott1964: 187). According to the convention used by Baumslag and Chandler(1968: 253), H_8 is isomorphic to $\mid X, Y, X^2, [X, Y^2]^2, Y^{-4}(XY)^4 \mid$

taken as the identity element. The Cayley diagram for this group is shown in Figure 1 which is adapted from d'Adhemar (1976:136). The double lines signify generator X and the single lines signify generator Y. Words consisting of any product or sequence of various powers of X and Y will then define particular classes of kin (including their equivalent siblings). The next step is to define a particular cluster of cousins as equal to spouse. Different spouse equations produce different structures of homogeneous redundancy. If the defining relations happen to be

$$X^2 = (X^{-1}Y^{-2}XY^2)^2 = I, \qquad Y^4 = (XY)^4$$

then there are four specific third cousins in the same class as BWsib (Lucich 1987: 146-8, 195). The redefinition of Figure 1 according to these relations[2] produces the group H_8 as a factor group of Figure 1. This is shown in Figure 2 as a central column surrounded by helical paths. Its vertices are labelled according to another factor group known as $32\Gamma_7a_1$.

This is produced when Y^4 is relabelled as the identity element, and the multiplicationtable for that group is in Thomas and Wood (1980).[3]

Generator X and Y can also permute so that the same structure may have two realisations. Realisation One has

$X = X^{-1}$ = ZHsib, Y = BC,

while Realisation Two has

$X = X^{-1}$ = BC, Y = ZHsib.

Realisation One has BWsib equivalent to FM(MBC)BDC and three other third cousins, while Realisation Two has BWsib equivalent to FMBSC (a second cousin) andFM(MBC)BDC plus others (Lucich1987: 195).

The helical structure can be opened out as shown in Figure 3, which as Realisation Two allows the allocation of kin terms from the Worora tribe

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2 The homomorphisms relevant to elementary kinship also include those between infinite groups and their factor groups, as illustrated in the relationships between 230 (infinite) space groups and the 32 (finite) crystallographic point groups (CoxeterandMoser1957: 35).

3 3 The factor group $32G_7a_1$ is produced by adjoining the relation $Y4^4 = I$.

The notation is from Hall and Senior(1964). This group has a factor group of structure C_ which is the homomorphic image produced when its subgroup $16G_2a_1$ is redefined as the kernel (Lucich1987:107).

of the Kimberley area of Western Australia (Lucich 1987: 330). The allocations are such that the inter-term relations correspond to appropriate inter-vertex relations on the model. For greater contrast, generator Y is shown as a dotted line.

Realisation One can serve as the scaffold for the kin terms of the Aluridja of South Australia (Elkin 1939: 210-32; 1940: 307, 344; Lucich 1987: 265-89). In other words the Worora and Aluridja systems can both be displayed as transforms of each other on the same group structure of H*. The distribution of terms on vertices is not exactly 1:1, and is subject to other specific rules.

The factor group $32\Gamma_7 a_1$ is also a factor group of the P_{44} wallpaper group where generator X stands for a half-turn, and generator Y stands for a quarter-turn. A visual realisation is shown in Figure 4 together with a P_2 subgroup realised as white arrows. There are only 17 possible wallpaper designs in the strict sense and some of their Cayley diagrams correspond to particular spouse equation structures. In this example the diagram for P_4 can be used to display $32\Gamma_7 a_1$ and the two kinship realisations as shownin Figures 5 and6. The H_8 or helical version in Figure 2 remains a more accurate model of spouse equivalence, since the planar format of P_4 separates particular cousins which are combined as spouses in the H^8 model.

Further, the P_2 subgroup of P_4 is the kernel of the C_homomorphic image corresponding to generation levels in Realisation One and patrimoieties in RealisationTwo. There is an asymmetry here in that

$$\frac{P_4}{P_2} = C_2, \text{ but } P_2 \times C_2 \neq P_4.$$

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[4] The defining relations for p4 are given by Coxeter and Moser (1957:46) as

$$X^2 = Y^4 = (YX)^4 = (XY)^4 = I$$

Factorgroup $32G_7a_1$ is produced by adjoining the relations $(YXY)^2(XY^2)^2 = I$.

The relevant C_2 factor group is produced by redefining the index 2 subgroup p2 as the kernel.

This suggests that sociocentric classes are more easily derived from kinship structures than the reverse relation. The planar realisation in P_4 is isomorphic to these kinship systems, provided that certain simplifying conditions are met, and the structural entailments allow the above speculations on the derivations of the two-class systems. In this view the elementary kinship structures are usefully seen as multiple superimposed homomorphic images. Developmental transformations can also be modelled through those factor groups which correspond to possible precursors. Finally, for each system the meaning of any element is ultimately definable by the combining of generators. Vertex 5 (for example) is XYX which is umari in Aluridja and ibaia in Worora. Translation then depends on appropriate substitutions for X and Y. The isomorphisms are summarised in Table I.

Table 1 Four realisations of P_4 and its P_2 subgroup

Cayley Diagram Figures 5,66	Wallpaper Figure 4	Aluridja Figure5	Worora Figure 6
Double line Generator X	Half turn	Marriage (ZHsib,BWsib)	Pastrifiliation (BC,Fsib)
Dotted line Generator Y	Quarter turn	Patrifiliation (BC°	Marriage (ZHsib)
Continous line Generator XY = Z	Displaced quarter turn	Matrifiliation (ZHsib)(BC) or ZC	Cross-generation affinal cycle (Fsib)(ZHsib) or FZH
Vertex	Equivalence class of -congruence motions -paths from reference	Equivalence class of -classificatory siblings -paths from ego	Equivalence class of
Vertex (XYX)		*umari* (ZDHsib)	*ibaia* (FZC)
P_2 subgroup 1 1x 8 8x 11 11x 14 14x 3 3x 16 16x 9 9x 6 6x (C_2 coset)	WHITE MOTIFS	OWN GENERATION LEVEL	OWN PATRIMOETY

The kinship entries in the table are an interchange or permutation of marriage and patrifiliation, and the fact that the Aluridja and Worora groups of operations can be precisely aligned or mapped onto each other illustrates one particular isomorphism.

It is also possible to select other generators such as XY or Z (which stands for matrifiliation in the Aluridja model).

The distribution of black and white in Figure 4 involves the same formal resource that allocates generation levels and patrimoieties to the kinship models for Aluridja and Worora respectively, as shown in the capitalised entries in Table I. In other words, the relation of homomorphism creates the reduced homomorphic image of structure C_2 which is variously realised as a regular two-colour contrast or the endogamous moiety system or the patrimoiety system.

Any axiomatic treatment must be broadened to include not only elementary kinship and wallpaper design but also other cultural systems of permutations, especially the equally tempered musical scale. Here the free group is C_∞ and the factor group is C_{12} which relations also correspond to the twelve hours on a clockface(Budden1972:429-31,436-8).

Table II from Lucich (1987: 438) summarises some of the isomorphisms.

Table 2 Strucutureof codes for different domains and purposes

DOMAIN	MEANS		END
	Group	Homomorphic Image	Purpose
Elementary kinship	$C_\infty \times D_1$, pl cm cm-col P_4 P_4g H_8	$C_4 \times C_2$, D_2, D_1 161_{2C1}, D_4 321_{7a1}	Balanced reciprocity, Align different systems, Reduction
Visual symetries	$C_\infty \times D_1$, pl cm P_4 P_4g C_∞ D_∞	C_n D_n	Express the code Enhance the code via colours
Equal temperament	C_∞	C_{12}	Align different keys and instruments
Kula	D_∞		Equalised reciprocity
Permutation systems		$(D_6$ D_4 D_2 $D_2 \times D_1$, $A_5)$	Systematisation

The table shows that the same purposes and structures can recur across entirely separate domains and cultures, and the first three entries display a common pattern of reduction to simple structures by using the homomorphisms of infinite groups.

The examples depend on homogeneous structures of equivalence classes, and the diverse manifestations are here reduced to their structures of combined operations. One necessary assumption is that the agents umversally and at some level are able to make computations isomorphic to relational products of the separate operations (Lehman 1985: 24-7, 40-1). It is further necessary that they can accomplish the appropriate homomorphic reductions.

The heading END/Purpose takes the argument from the formal to the explanatory. Isomorphisms emerge when agents' purposes imply reciprocity, alignment or systematisation, when the domains require regular, successive, reversible and coordinated transitions, and when the mind is capable of formal operational thought. The latter point refers to Piaget's assumptions about combinatorial thought, human cognitive capacities and their realisation in socially shared scliemata. However, it is not mind alone which is the cause here, otherwise the manifestations would be everywhere.

Table III (Lucich 1987: 445) shows the distribution of those domains which become the vehicle for intended homogeneous structures of cognitive operations.

Table 3 Distribution of combinatorial sign systems by domain and societal type

Domain	Hunter-gatherer	Communal	Theocratic	Civilizational
Elementary kinship	+++	+	-	
Visual symmetries	+	++	+++	
Equal temperament				+++
Permutation systems	++	+	+	+

The societal typology shouldbe regarded as averybroad classification based on technology and scale, and the table entries are only approximate measures of emphasis. The point of the table is that it is the varying social context (and not just the mind) which calls forth the appropriate purposes.

An explanatory synthesis therefore entails the complementary claims of mind and social exigency. Under certain preconditions, practices with symmetries of combined operations are created to achieve particular purposes. Their associated sign systems are restricted in the combinations of their elements, with practical limits expressible finally in terms of logical consistency.

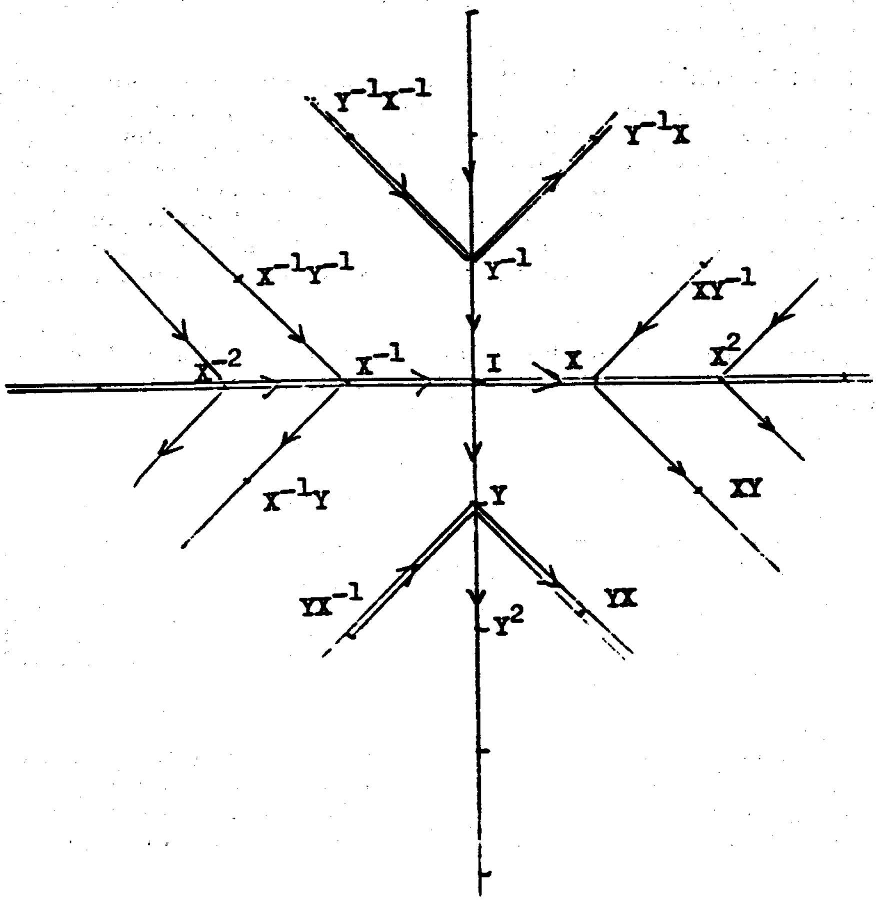

Figure 1 Free group G generate by { x, y }

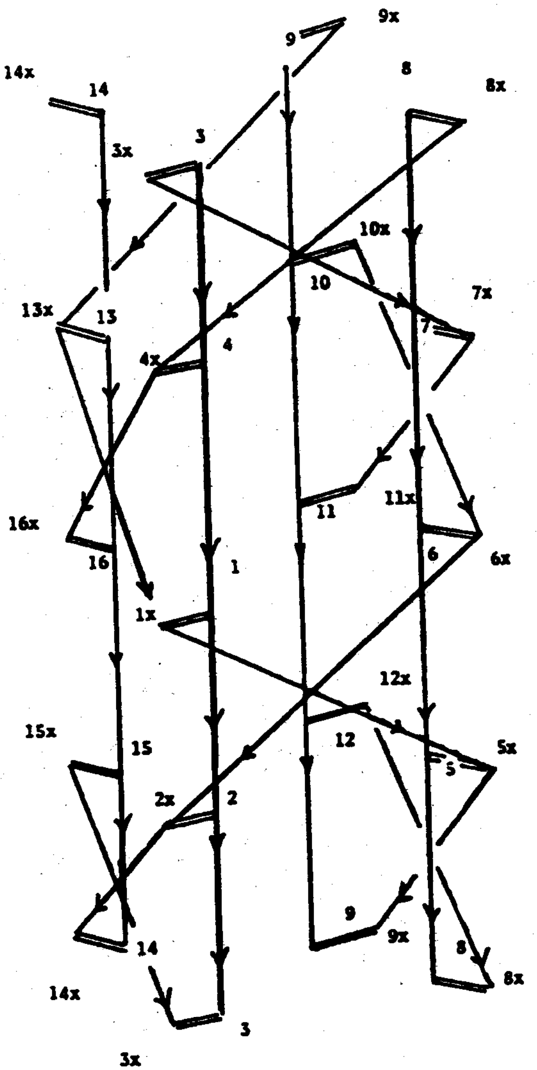

Figure 2 Cayley diagram of H_8 with vertex labels from its 321_7a_1 factor group

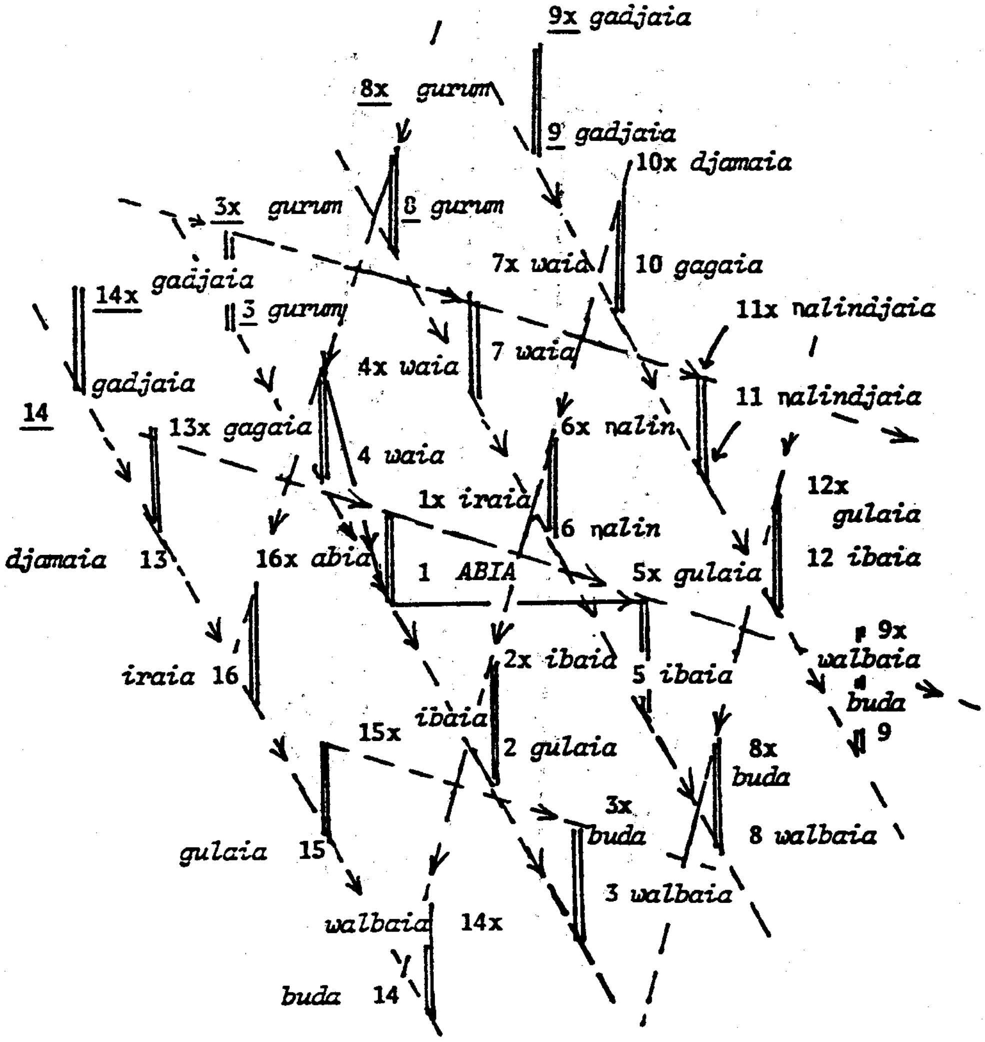

Figure 3 Realisation three of H_8 with Worora kin categories and with numbers the 321_7a_1 factor group

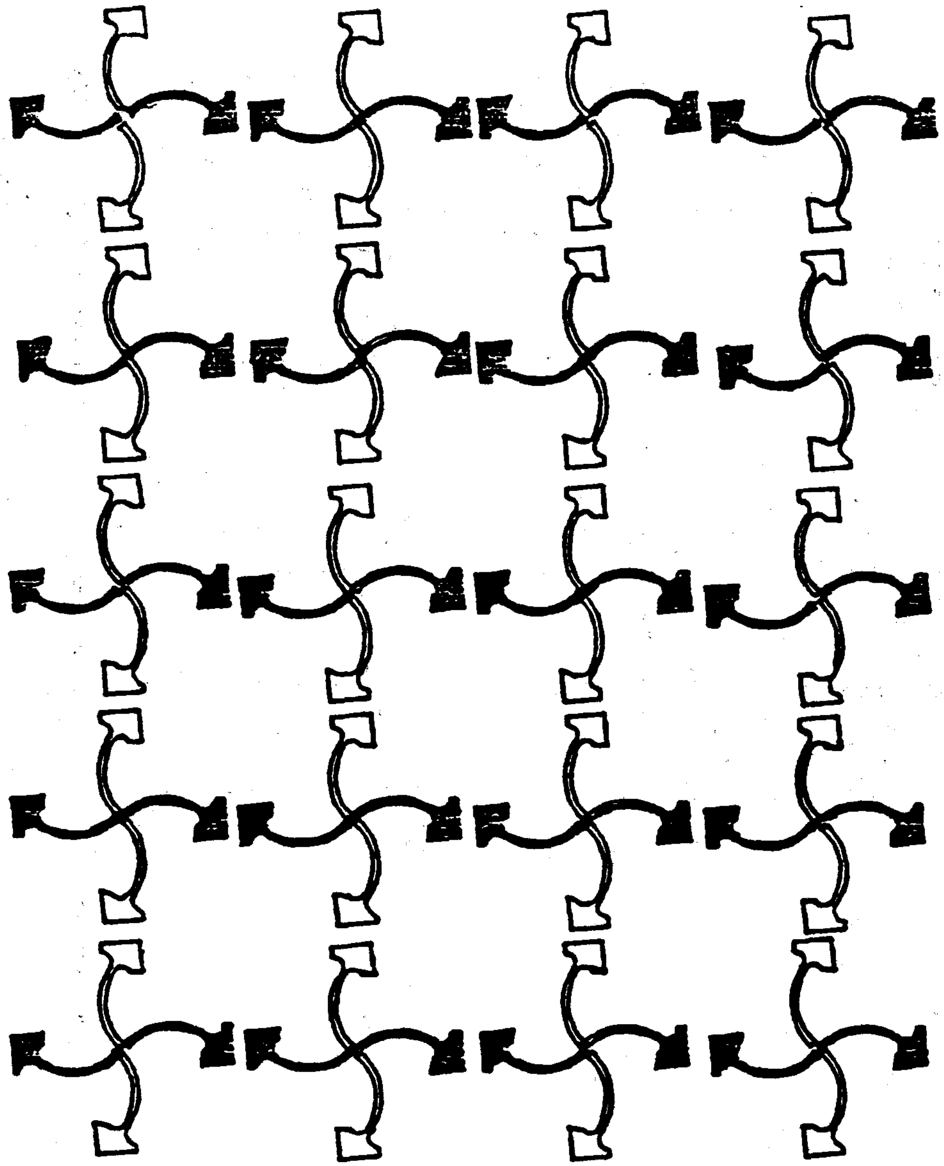

Figure 4 The P_2 subgroup of P_4 realised as the kernel of the latter's two-colour contrast

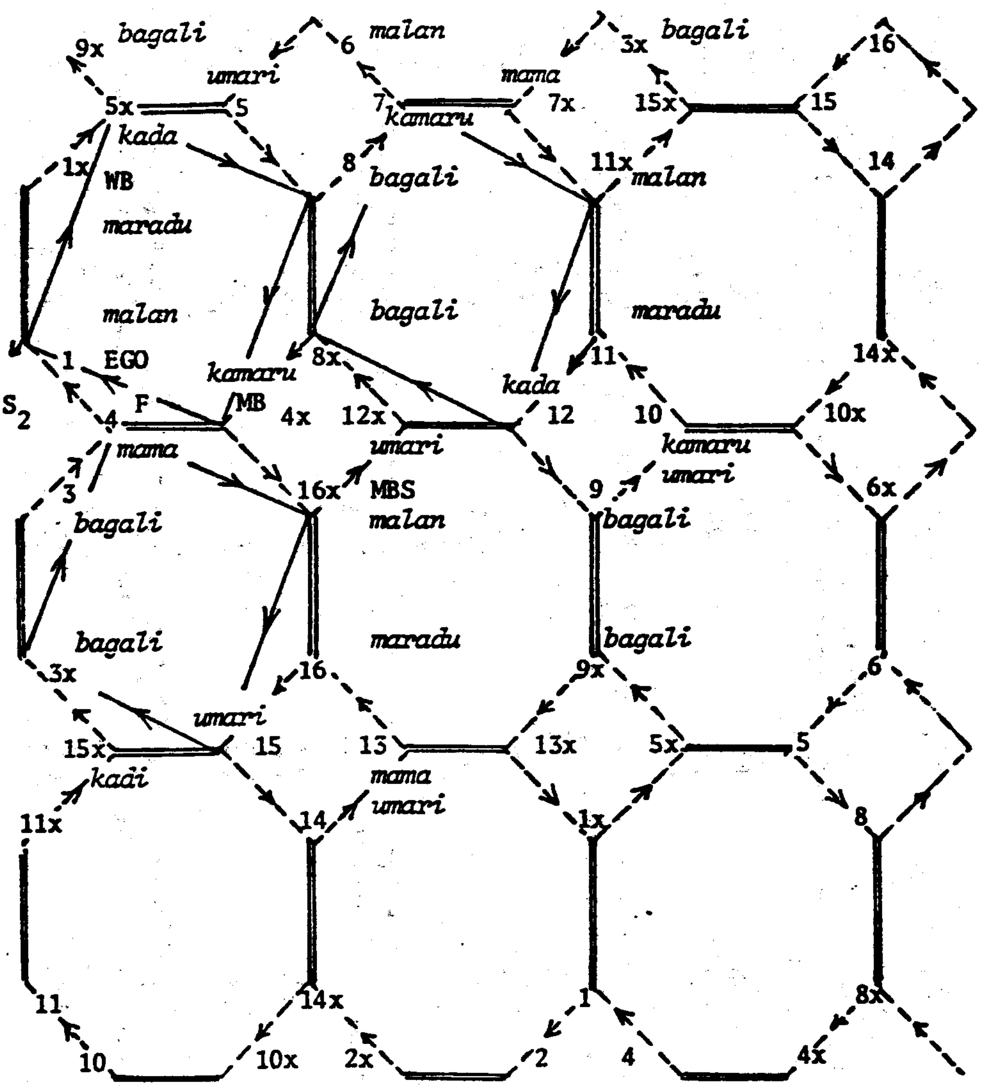

Figure 5 Realisation T One of P_4 with vertices labelled according to 321_7a_1 and Aluridja kinship

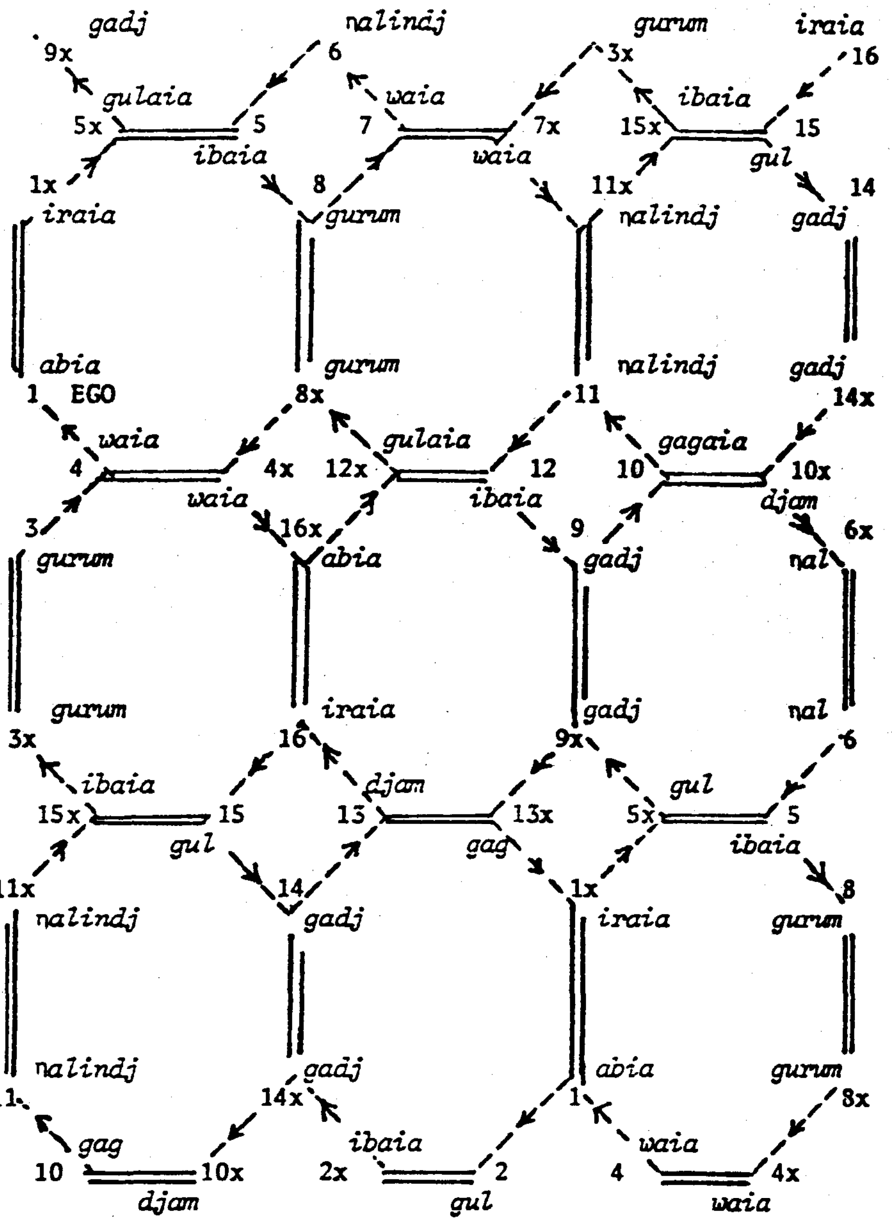

Figure 6 Realisation Two of P_4 with vertices labelled according to 321_7a_1 and Worora kinship

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