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Comment on "An improved fractal equation for the soil water retention curve" by E. Perfect et al.

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We wish to point out a significant conceptual error in the derivation of a fractal model water retention curve by *Perfect et al.* [1996]. Their derivation is based on the properties of the Menger sponge (see, e.g., Figure 1b in the work by *Rieu and Sposito* [1991a]), a well-known fractal model for porous media that exhibit length-scaling invariance [*Mandelbrot*, 1983, p. 134]. A crucial step in the derivation presented by *Perfect et al.* [1996, equation (6a)] is their decision to equate the volumetric water content ($\theta_{i,j}$) with the subunit porosity ($\Phi_{i-1,j}$) at one hierarchical level below level i, after the same number of iterations j of the generator. Their postulate is thus:

$$\theta_{i,j} = \Phi_{i-1,j} \equiv 1 - (b^{-i+j+1})^{D-3} \qquad 1 \le i \le j \tag{1}$$

where D is the fractal dimension of the sponge and b is a scale factor which relates the length of a side of the (cubic) initiator of a Menger sponge to the length of a side of the (cubic) subunit created by the generator of the sponge. (See, e.g., work by *Mandelbrot* [1983, p. 134] and *Perfect et al.* [1996] for the details of Menger sponge generation.) We believe that the first equality in (1) is incorrect, for the following reasons.

The porosity $\Phi_{i-1,j}$ in (1) applies to the volume of pores contained in (cubic) subunits whose sides scale as b^i with that of the initiator and whose total volume is V_{SUB}:

$$V_{\mathrm{SUB}}\Phi_{i-1,j} = \text{volume of pores} \tag{2}$$

By contrast, the volumetric water content $\theta_{i,j}$ applies to the same volume of pores but as contained in the entire Menger sponge, whose volume is V_{T}:

$$V_{\mathrm{T}}\theta_{i,j} = \text{volume of pores} \tag{3}$$

The ratio $(V_{\mathrm{SUB}}/V_{\mathrm{T}})_i$ of the total volume of porous subunits at the $i - 1$ hierarchical level to the total volume of the sponge thus connects $\theta_{i,j}$ to $\Phi_{i-1,j}$ at any hierarchical level i, after j iterations of the generator:

$$\theta_{i,j} = (V_{\mathrm{SUB}}/V_{\mathrm{T}})_i\Phi_{i-1,j} \qquad 1 \le i \le j \tag{4}$$

The proportionality factor in (4) can be calculated readily by induction. At hierarchical level $i = 1$ the Menger sponge has been divided into b^3 (cubic) subunits (each of whose sides is of length $1/b$ relative to the length l of a side of the (cubic) initiator), and n of these subunits have been retained to form the solid matrix of the sponge [*Perfect et al.*, 1996]. Thus, when $i = 1$, $(V_{\mathrm{SUB}}/V_{\mathrm{T}})_1 = n/b^3$ $(1 \le n \le b^3)$. At level $i = 2$ the n subunits of level $i = 1$ are again divided into b^3 new subunits, and n of these new subunits are retained to form the solid matrix. The ratio $(V_{\mathrm{SUB}}/V_{\mathrm{T}})_2$ is thus equal to $(n/b^3)^2$. Thus, at any level $i \ge 0$, $(V_{\mathrm{SUB}}/V_{\mathrm{T}})_i = (n/b^3)^i$. Therefore (4) can be expressed as

$$\theta_{i,j} = (n/b^3)^{i-1}\Phi_{i-1,j} \qquad 1 \le i \le j \tag{5}$$

Equation (1), in contrast with (5), implies tacitly that the total volume of the sponge is simply equal to that of its solid-matrix subunits, a condition that is met only for the initiator before the removal of $(b^3 - n)$ subunits, at a "zero level" of the hierarchy (i.e., set $i = 1$ in (5)).

Perfect et al. [1996] employed (1) instead of (5) to derive a model water retention curve that turned out to have the same mathematical form as an empirical equation proposed by *Ross et al.* [1991]. The model water retention curve that instead follows from (5) can be found by combining (5) with the accepted definition of $\Phi_{i-1,j}$ (second equality on the right side of (1)) and the well-known expression for the fractal dimension of a Menger sponge [*Mandelbrot*, 1983, p. 134],

$$n = b^D \tag{6}$$

Therefore

$$\begin{aligned}\theta_{i,j} &= (b^D/b^3)^{i-1}[1 - (b^{-i+j+1})^{D-3}]\\ &= (b^{i-1})^{D-3}[1 - (b^{-i+j+1})^{D-3}]\\ &= (b^{i-1})^{D-3} - (b^j)^{D-3}\end{aligned} \tag{7}$$

An expression for the water retention curve is then derived by noting that

$$\Phi_{0,j} = 1 - (b^j)^{D-3} \tag{8}$$

is the porosity of the Menger sponge [*Rieu and Sposito*, 1991a; *Perfect et al.*, 1996] and that

$$b^i \propto \psi_i \tag{9}$$

where ψ_i is the water potential that is sufficient to empty pores created at hierarchical level i [*Perfect et al.*, 1996]. Introduction of (8) and (9) into (7) produces the fractal model water retention curve:

$$\theta_{i,j} = (\psi_i/\psi_1)^{D-3} + \Phi_{0,j} - 1 \tag{10}$$

where $\psi_1 \propto b$ is the water potential that is sufficient to empty only the largest pores (i.e., set $i = 1$ in (9)). We note in passing that $\theta_{i,j}$, the "residual" volumetric water content achieved after j iterations at hierarchical level j, can equal zero only if the scale factor $b^j \uparrow \infty$ when $i = j$ in (7) or, equivalently, if the water potential $\psi_j \uparrow \infty$ when $i = j$ in (10). In the context of a fractal approach this means that infinitely small pores are assumed to occur, and the solid phase can vanish, leaving the fractal model to represent solely the pore space [*Perrier et al.*, 1996]. In this case (8) requires $\Phi_{0,j} = 1$, and (7) or (10) becomes equivalent to the fractal model of *Tyler and Wheatcraft* [1990].

Equation (10) previously was derived and tested successfully

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by *Rieu and Sposito* [1991b] on the basis of their fractal porous medium model [*Rieu and Sposito*, 1991a]. It is not the same as the empirical equation of *Ross et al.* [1991]. That (10) results from (5) is not surprising, since the porous medium model of *Rieu and Sposito* [1991a], like the Menger sponge, imparts fractal character to both the pores and the solid matrix. The latter is represented by the nonporous subunits that are retained at the level j of the hierarchy, when the iteration process is ended because of the typical existence of a lower bound on the range of length scales over which fractal properties are exhibited by a porous medium. *Perrier et al.* [1996] have shown that if no assumption is made about solid matrix geometry, the general equation for the water retention curve in a soil exhibiting a fractal pore size distribution is

$$\theta_{i,j} = A[(\psi_i/\psi_1)^{D-3} - 1] + \Phi_{0,j} \tag{11}$$

where the parameter A represents the largest value possible for the fractal porosity. We can readily verify that (10) is the particular case of (11) wherein A is set equal to 1.

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Reply

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We appreciate the comments by *Bird* [this issue] and *Perrier et al.* [this issue] pointing out an error in the derivation of (12) of *Perfect et al.* [1996]. As these authors discuss, this error was the result of confusion between the porosity (and thus volumetric water content) for a single cubic subunit at the ith level after j iterations of the Menger sponge generator and the ith level porosity (and thus ith level volumetric water content) for the entire Menger sponge after j iterations of the generator. Having acknowledged this error, we will now proceed to rederive (12) of *Perfect et al.* [1996] using the correct expression for the ith level volumetric water content and to explore some of the physical implications of the resulting equation.

From (7) of *Perrier et al.* [this issue] the ith level volumetric water content after j iterations of the Menger sponge generator, $\theta_{i,j}$, is given by

$$\theta_{i,j} = (b^{i-1})^{D-3} - (b^j)^{D-3} \quad (1)$$

where b is the scaling factor, D is the mass fractal dimension, and $1 \leq i \leq j$ is the number of iterations of the generator. The saturated water content, $\theta_{1,j}$, is obtained by setting $i = 1$ in (1), that is,

$$\theta_{1,j} = 1 - (b^j)^{D-3} \quad (2)$$

Assuming the smallest voids do not drain, the residual water content, $\theta_{j,j}$, can be obtained by setting $i = j$ in (1), that is,

$$\theta_{j,j} = (b^{j-1})^{D-3} - (b^j)^{D-3} \quad (3)$$

Following *Brooks and Corey* [1964], we can define an effective saturation, S_e, for the Menger sponge as

$$S_e = (\theta_{i,j} - \theta_{j,j})/(\theta_{1,j} - \theta_{j,j}) \quad (4)$$

By substituting (1), (2), and (3) into (4), we obtain the following expression for the effective saturation as a function of b^i:

$$S_e = [(b^i)^{D-3} - (b^j)^{D-3}]/[(b^1)^{D-3} - (b^j)^{D-3}] \quad (5)$$

The b^i is related to the soil water tension at the ith level, ψ_i, by [*Perfect et al.*, 1996]

$$b^i = k\psi_i \quad (6)$$

where k is a constant. Finally, substituting (6) into (5) yields

$$S_e = [(\psi_i)^{D-3} - (\psi_j)^{D-3}]/[(\psi_1)^{D-3} - (\psi_j)^{D-3}] \quad (7)$$

where ψ_1 is the tension that drains the largest voids and ψ_j is the tension that drains the smallest voids.

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Recall that the *Ross et al.* [1991] empirical equation for the soil water retention curve is given by

$$S = (\psi^{-c} - \psi_d^{-c})/(\psi_a^{-c} - \psi_d^{-c}) \quad (8)$$

where S is relative saturation, ψ_a is the air entry value, ψ_d is the tension at dryness, and c is a constant. Assuming $S_e \approx S$ (i.e., $\theta_{j,j} \rightarrow 0$ in (4)), then (7) is equal to (8), and the following relations hold between the parameters of the two equations:

$$D = 3 - c \quad (9a)$$

$$\psi_1 = \psi_a \quad (9b)$$

$$\psi_j = \psi_d \quad (9c)$$

These equivalencies are the same as those obtained by *Perfect et al.* [1996], with the important exception that $D = 3 - c$ and not $3 + c$. If we substitute our original estimates for c into (9a), we obtain a range of 2.91–4.37 for the fractal dimension, with 91% of the D values > 3.

We have shown the fractal nature of (8) is still valid. Thus all of the nonlinear fits and estimates of ψ_1 and ψ_j by *Perfect et al.* [1996] are also still valid. The estimates of D in Tables 4–6 of *Perfect et al.* [1996], however, are incorrect. The correct values can be obtained, as noted by *Bird* [this issue], with the simple transformation $D = 6 - D_{(14a)}$, where $D_{(14a)}$ is the fractal dimension calculated from c using the erroneous equation (14a) of *Perfect et al.* [1996]. This transformation results in a negative correlation between the fractal dimension and porosity, as observed by *Bird* [this issue].

Values of $D > 3$ are not physically meaningful. They may be caused by fitting over the range $0 \leq \psi \leq 1.5 \times 10^3$ kPa, instead of over the entire range of ψ down to oven dryness (E. Perfect, Estimation of soil mass fractal dimensions from water retention data, submitted to *Water Resources Research*, 1997). It is also possible that (7) is not a good theoretical model for the water retention properties of natural porous media or that some of the assumptions used in its derivation are not valid.

In deriving (7) it was assumed that b^i can be related to ψ_i by (6), the well-known Young-Laplace equation. While this equation is commonly assumed to hold over the entire range of tensions employed [e.g., *Tyler and Wheatcraft*, 1990; *Rieu and Sposito*, 1991], it is probably not valid at very large tensions, when water exists mainly as thin films on void surfaces. Under these conditions the b^i and ψ_i are better related by [*Toledo et al.*, 1990]

$$b^i = q(\psi_i)^{1/m} \quad (10)$$

where q is a constant and m depends on the nature of the forces responsible for the solid-liquid interaction.

Substituting (10) into (5) and comparing the result with (8), yields (9b), (9c), and

$$D = 3 - mc \quad (11)$$

Toledo et al. [1990] report that $1 \leq m \leq 3$, with $m = 0.48$ for their data. Using this value of m, along with the estimates of c from *Perfect et al.* [1996], in (11) results in values of D much

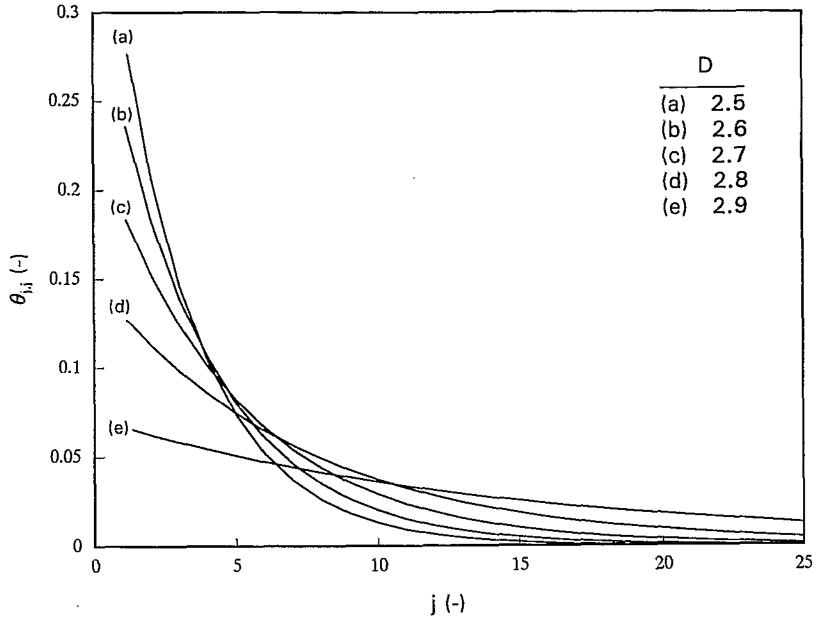

Figure 1. Residual water content of the Menger sponge, predicted using (3), as a function of j and D when $b = 2$.

closer to 3. However, m would need to be negative for the majority of the D values to be less than 3. Since there is neither theoretical nor experimental support for $m < 0$, we conclude that values of $D > 3$ were not a result of using (6) instead of (10).

The equivalencies in (9) assume that the residual water content of the Menger sponge is so small that it is negligible. While this is generally true, the actual magnitude of the residual water content depends on j, b, and D as described by (3). For any given value of $b > 1$, $\theta_{j,j} \rightarrow 0$ as $j \rightarrow \infty$ (Figure 1). Although the predicted curves are plotted as continuous functions, the discrete nature of the voids in the Menger sponge model actually produces a stepwise function. Note that $\theta_{j,j} \rightarrow 0$ more slowly as $D \rightarrow 3$ (Figure 1). As a result, sponges with fractal dimensions only slightly less than 3 can have significant residual water contents even after many iterations of the generator. Thus the assumption that $S_e \approx S$ may not always be valid.

An alternative approach is to derive a fractal water retention curve explicitly for S instead of for S_e. This can easily be done by dividing (2) into (1) and then substituting (6) into the result, yielding

$$S = [(\psi_i/\psi_1)^{D-3} - (b^j)^{D-3}]/[1 - (b^j)^{D-3}] \tag{12}$$

Equations (12) (above) and (10) (from *Perrier et al.* [this issue]) are both based on the Menger sponge model and use the same assumptions. Therefore they should be different forms of the same equation. This can be seen by combining (7)–(9) of *Perrier et al.* [this issue] to predict relative saturation, S, instead of volumetric water content, $\theta_{i,j}$. The resulting equation is identical to (12).

Equation (12) was fitted to the Elora data of *Perfect et al.* [1996]. The estimates of b^j ranged from 1.2×10^4 to 7.6×10^5, and were linearly related to ψ_j from (8) by the following regression equation ($R^2 = 0.47$):

$$b^j = 6.14\psi_j \tag{13}$$

The slope in (13) is equivalent to the constant k in (6). Goodness-of-fit statistics for (12) were identical to those for (8), as were the estimates of ψ_1 and D. Since (8) and (12) gave the same values of D, we conclude that the approximation $S_e \approx S$ was not responsible for estimates of $D > 3$.

It appears from the above discussion that estimates of $D > 3$ may be real for some soils. Such media cannot be considered mass fractals. We propose that the *Ross et al.* [1991] function be used to discriminate between those media that are fractal and those that are not. Equation (8) provides a good fit to a wide range of experimental water retention data (Figure 2). The retention curves predicted by (8) are convex for negative values of c, concave for positive values of c, and approach linearity on the semilog scale as $c \rightarrow 0$. We attempted to fit (11) of *Perrier et al.* [this issue] to the water retention data in Figure 2, but it failed to converge uniquely in both cases. *Perrier et al.* [1996] encountered similar problems with this equation.

In the case of nonfractal porous media, (8) must be viewed as entirely empirical. The c parameter will be negative and the water retention curve will be convex when plotted on a semilog scale. A potentially useful version of (8) can be obtained for structured or macroporous media by assuming that $\psi_a \rightarrow 0$. Since $c < 0$, this assumption results in the following relation:

$$S = 1 - (\psi/\psi_d)^{-c} \tag{14}$$

Equation (14) is shown in Figure 2 fitted to the convex data set. The more negative estimate of c obtained using (14), as compared to (8), is probably attributable to the relatively poor fit of (14) close to saturation (Figure 2). Keep in mind that neither exponent can be interpreted as a fractal dimension using (9a) above.

In the case of fractal porous media, c in (8) will be positive and the water retention curve will be concave when plotted on a semilog scale. Assuming $c > 0$ and $\psi_d \rightarrow \infty$, (8) simplifies to

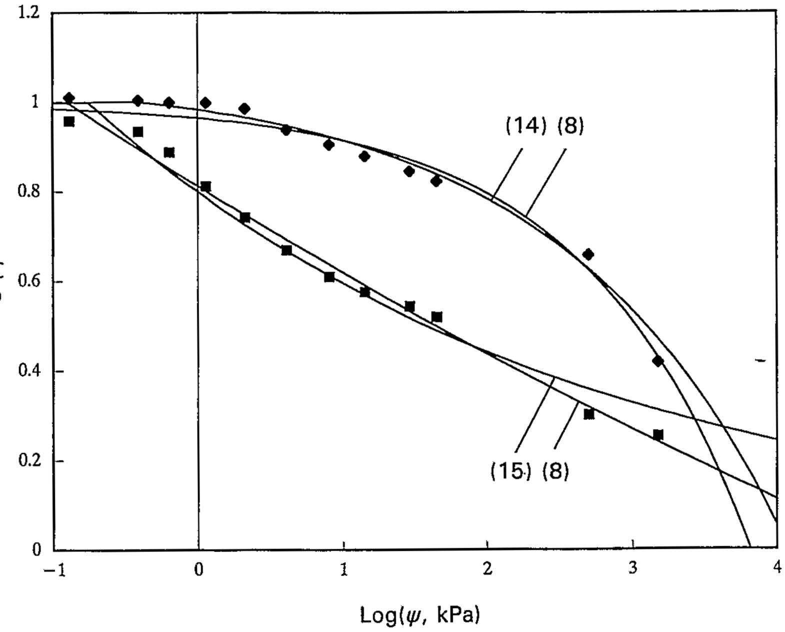

Figure 2. Observed and predicted water retention curves for two soil cores from the Elora data set of *Perfect et al.* [1996] yielding the maximum and minimum estimates of c. Numbers in parenthesis refer to the equations in this reply. Parameter estimates and residual sums of squares (RSS) were: $\psi_a = 4 \times 10^{-1}$ kPa, $\psi_d = 1 \times 10^4$ kPa, $c = -0.28$, and RSS = 0.005 for (8) (convex curve); $\psi_d = 7 \times 10^3$ kPa, $c = -0.38$, and RSS = 0.008 for (14); $\psi_a = 1 \times 10^{-1}$ kPa, $\psi_d = 5 \times 10^4$ kPa, $c = 0.03$, and RSS = 0.007 for (8) (concave curve); and $\psi_a = 2 \times 10^{-1}$ kPa, $c = 0.13$, and RSS = 0.019 for (15).

$$S = (\psi/\psi_a)^{-c} \tag{15}$$

which is equivalent to the *Brooks and Corey* [1964] equation with $S = S_e$ and $\lambda = c = 3 - D$. The goodness of fit of (15) as compared to (8) is illustrated in Figure 2. Equations (8) and (15) give different estimates of c, as noted by *Ross et al.* [1991]. This means that $D = 2.97$ for (8), while $D = 2.87$ for (15). The lower value of D for (15) as compared to (8) is probably attributable to the relatively poor fit of (15) at very high tensions (Figure 2).

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